Assignment Problems

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I. Introduction

An assignment problem is one in which a number of goods, each in some fixed quantity, must be assigned to a number of individuals. The class of assignment problems that will concern us here are those in which no monetary transfers are possible. Assigning committee positions to members of Congress or dormitories to students are but two of many such examples.

When the individuals’ tastes are known, it is not difficult in principle to achieve an assignment of goods to individuals that is Pareto efficient. But this becomes considerably more difficult when preferences are private information because one must then ensure that no individual has any incentive to misreport his or her preferences.

In a seminal _JPE_ paper, Hylland and Zeckhauser (1979) consider assignment problems in which each individual can receive at most one good and at most one unit of it (as in the two examples above). They showed that if individuals are endowed with fiat money and participate in a market that sets nominal prices for the probabilities with which goods can be obtained, then competitive equilibrium prices (for probabilities) exist and yield ex ante efficient lotteries that can be resolved to produce ex post efficient outcomes. Consequently, when there are sufficiently many individuals so that no single individual has any significant impact on prices, each individual would be willing to report his preferences truthfully in a mechanism that computes and implements the competitive equilibrium outcome for those preferences. Such is the mechanism proposed by Hylland and Zeckhauser.

An even more challenging class of assignment problems are the so-called combinatorial assignment problems. In such a problem, there are...
again many units of many goods to be allocated, but there are no a priori restrictions on the bundles of goods that individuals can receive. Especially challenging are cases in which individual preferences over bundles of goods exhibit complementarities.

Recently, an important combinatorial assignment problem has been considered by Budish (2011). He considers the challenging problem of assigning classes to students, a problem in which complementarities arise naturally from course scheduling constraints even if student preferences over classes, without those constraints, are additively separable. 3

As in the Hylland-Zeckhauser model, the goods in Budish’s (2011) model, namely classes, are indivisible. Hylland and Zeckhauser (1979) circumvent the indivisibility problem by creating a market for probabilities. Unfortunately, as Budish observes, in the presence of complementarities there may be no prices for the probabilities with which individual classes can be obtained that lead students to choose lotteries over bundles of classes that efficiently exhaust the total available probability and that are feasible to carry out. In short, the combinatorial assignment problem cannot, in general, be solved by using the lottery technique of Hylland and Zeckhauser. One must deal with the indivisibilities and complementarities head on.4

Like Hylland and Zeckhauser (1979), Budish (2011) uses a market mechanism with fiat money to attack the problem. Students are given “income” in the form of fiat money that can be used to purchase classes at prices that are determined in market equilibrium. Importantly, Budish allows the students’ preferences to be almost completely general and does not assume that the fiat money has any intrinsic value to them (unlike other mechanisms in use for this problem).

Because issues such as “fairness” are particularly important in this and other assignment contexts, it would be natural for each student to receive the same amount of fiat money. However, as Budish (2011) shows, this can lead to the nonexistence of the type of market equilibrium that he considers. Budish’s striking result is that, with arbitrarily small departures from equal relative incomes, existence is restored, and several attractive efficiency and fairness criteria can be obtained. Furthermore, with large numbers of individuals, Budish’s market mechanism is approximately incentive compatible.

The objective in this short note marking the 125th anniversary of the Journal of Political Economy is modest. It is shown here that Budish’s (2011) result can be generalized to allow arbitrary preferences and both

3 I am grateful to Eric Budish for pointing this out.
4 But see Budish et al. (2013) for particular conditions under which the lottery technique can be made to work.
divisible and indivisible goods, as would exist, for example, in the committee assignment problem when the workload on some committees can be divided up in any way among committee members. Thus, while the absence of divisible goods is important in Budish’s proof, it is inessential for his results.

On the technical side, the proof offered here is rather simple. Indivisibilities create discontinuities in demand as prices vary, because strictly preferred bundles can suddenly become affordable. These discontinuities are the source of most of the complications that arise in Budish’s (2011) clever proof. The main technical contribution here is to note that one can avoid discontinuities altogether by considering a surrogate economy in which agents, instead of maximizing utility subject to their budget constraint, maximize a Lagrangian in which violations of their budget constraint yield a suitably high utility cost per unit of overexpenditure. Equilibria of this surrogate economy are shown to yield equilibria in the sense of Budish’s paper. It is entirely possible that this surrogate economy, because it is continuous, might lead to more efficient and/or stable algorithms for computing the requisite equilibria. But these computational issues have not been explored here in any detail whatsoever.

II. Assignment Problems

An assignment problem, \((I, L, X, u, \omega)\), consists of the following items:

1. \(I\) and \(L\) are positive integers, where \(I\) is the number of agents and \(L\) is the number of commodities;
2. \(X = \times_{i=1}^I X_i\), where each consumption set \(X_i\) is a compact set of non-negative vectors in \(\mathbb{R}^L\) with \(0 \in X_i\);
3. \(\omega\) is an aggregate endowment vector in \(\mathbb{R}^L\) such that \(\omega_l > 0\) for every \(l = 1, \ldots, L\);
4. \(u = (u_1, \ldots, u_I)\), where each utility function \(u_i : X_i \rightarrow [0, 1]\) is continuous.\(^7\)

\(^5\) Budish (2011) allows arbitrary preferences except for the assumption that no agent is indifferent between any two bundles in his finite consumption set. Because divisible goods are allowed here, my consumption sets can be uncountably infinite. Therefore, to accommodate continuous preferences, indifference must be permitted, and this is done whether consumption sets are finite or infinite.

\(^6\) The proof technique employed here can also provide a generalization of the results of Dierker (1971) to include both indivisible and divisible goods, while at the same time simplifying his proof.

\(^7\) All the results can be derived under the slightly more general assumption that for each consumer \(i\) there is a reflexive and transitive (but possibly incomplete) binary relation, \(\succsim_i\), on \(X_i\) that is continuous; i.e., for every \(x \in X_i\), the sets \(\{y \in X_i : y \succsim_i x\}\) and \(\{y \in X_i : x \succsim_i y\}\) are closed.
Remark 1. Consumer i’s consumption set $X_i$ need not be convex or even connected. In particular, an assignment problem here can accommodate the simultaneous presence of divisible and indivisible goods.

For any $p \in [0, \infty)^L$, for any $c_i > 0$, and for any agent $i$, let $D_i(p, c_i) \subseteq X_i$ be the set of solutions to the maximization problem,

$$\max_{x \in X_i} u_i(x) \text{ subject to } px_i \leq c_i.$$ 

Let $\|\cdot\|$ denote the Euclidean norm. For any $\varepsilon > 0$ and for any $c_1, \ldots, c_I > 0$, let $\varepsilon = (c_1, \ldots, c_I)$ and define

$$\delta_i(p, \varepsilon) = \sup \|x_i - y_i\|,$$

where the supremum is over all agents $i \in \{1, \ldots, I\}$ and all $x_i, y_i \in \bigcup_{i \in \mathbb{I}} D_i(p, c_i + \varepsilon')$.

Throughout the paper, for any $l \in \{1, \ldots, L\}$, $e_l = (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes the $l$th unit vector in $\mathbb{R}^L$.

The main result below replicates the main result in Budish (2011), but does not require the sets $X_i$ to be finite, requiring instead only that they be compact. Also included is the minor modification that the positive incomes $c_1, \ldots, c_I$ of the agents can be arbitrary, whereas Budish focuses on the most central case in which $c_1 = \cdots = c_I = 1$.

Theorem 1. For any assignment problem $(I, L, X, u, \omega)$, for any $\varepsilon > 0$ and for any positive real numbers $c_1, \ldots, c_I$, there exist $p^* \in [0, \infty)^L$ and $x^* \in X$ such that

a. for every agent $i$,

i. $p^* x_i^* \leq c_i + \varepsilon$, and

ii. $x_i^*$ solves $\max_{y \in X_i} u_i(y)$ subject to $p^* y_i \leq \max(p^* x_i^*, c_i)$;

b. $\|z^*\| \leq \delta_i(p^* c_i) \sqrt{L}/2$, where $c = (c_1, \ldots, c_I)$ and where for each $l = 1, \ldots, L$,

$$z_i^* = \begin{cases} \sum_i x_i^* - \omega_l, & \text{if } p_i^* > 0 \\ \max \left(\sum_i x_i^* - \omega_l, 0\right) & \text{if } p_i^* = 0; \end{cases}$$

and

c. $p^* \omega \leq \sum_i (c_i + \varepsilon)$.

The present result is not an exact replication of the result in Budish (2011) because the bound on the market-clearing error here (specifically, the coefficient of $\sqrt{L}/2$ in part b of theorem 1) can be smaller or larger than the bound that he describes in his n. 15. However, the important feature of both bounds is that they are of the same order of magnitude and, most importantly, that they are independent of the number of agents, $I$. 
Remark 2. To obtain the result in Budish (2011), restrict each $X_i$ to be a finite set of nonnegative vectors with integer coordinates, set $c_i = \cdots = c_I = 1$ above, and define his $b^*_i = \max(p^* x^*_i, 1)$ for every $i = 1, \ldots, I$. Note that theorem 1 cannot be obtained from Budish’s result by discretizing the compact $X_i$ with a finite grid and then taking the limit as the grid size shrinks to zero. Such a limiting argument can ensure in part $o(ii)$ only that $x^*$ solves $\max_{y \in X_i} u_i(y)$ subject to $p^* y < \max(p^* x^*_i, c_i)$, because strictly better bundles that are unaffordable along the sequence might become exactly affordable at the limit.

Remark 3. Endowing the agents with even slightly different amounts of fiat money, instead of with real goods, and allowing the vector of goods prices to be any nonnegative vector, including the zero vector, means that differences in real incomes can be arbitrarily large and will be determined by the goods prices in equilibrium. This can be advantageous since efficiency might sometimes require such real income differences when preferences are not strictly monotone (e.g., as in the problem of assigning classes to students).

Budish (2011) introduces two fairness-related concepts that can be usefully applied to problems that include indivisible goods. The first of these is an agent’s “maxmin utility.” Before defining this, first define, for any positive integer $n$, agent $i$’s $n$-maxmin utility to be the utility number $\max \min(u_i(y_1), \ldots, u_i(y_n))$, where the maximum is over all $y_1, \ldots, y_n \in X_i$ such that $\sum_{j=1}^n y_j \leq \omega$. Then, define agent $i$’s maxmin utility to be his $I$-maxmin utility. Maxmin utility is one way to generalize the “I cut you choose” method of fair division to many agents. Budish’s second fairness concept presumes that $X_i$ contains only vectors with integer coordinates and is as follows. An allocation $x \in X_i$ is envy-free up to a single unit of a single good iff for every pair of agents $i$ and $j$, if $u_i(x_j) > u_i(x_i)$, that is, if $i$ envies $j$, then there is a commodity $l$ such that either $u_i(x_j) \geq u_i(x_i - e_i)$ or $x_j - e_i \notin X_i$.

Budish (2011) shows that the allocations that he obtains are envy-free up to a single unit of a single good and that, while they might not yield each agent his maxmin utility, they do yield each agent his $(I + 1)$-maxmin utility. Budish also shows that his allocations cannot be Pareto improved on if the agents are allowed to trade among themselves after the assignment is made. Budish does not rule out the possibility that if there is ex-

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9 Relative incomes can be made arbitrarily similar. With $c_i = \cdots = c_I = 1$, the ratio of any pair of the incomes $b^*_i, \ldots, b^*_I$ defined in the previous remark is between 1 and $1 + \epsilon$.

10 Indivisibilities can lead to nonexistence of efficient and envy-free allocations. See Budish (2011).

11 Budish (2011) uses the term maxmin share since he focuses on the bundle that achieves the maxmin utility. I find it more convenient to define the utility level, even though this number obviously depends on the particular utility representation.
cess supply, then that excess supply could be used to achieve a Pareto improvement.

The remarks to follow, in part, describe the sense in which Budish’s important fairness and efficiency results can be maintained and sometimes slightly improved on. In these remarks, \( \hat{p} \) and \( \hat{x} \) are as in theorem 1.

**Remark 4 (\( \eta \)-envy-free).** If \( \epsilon_1 = \cdots = \epsilon_I = 1 \), then for any \( \eta > 0 \), there is \( \epsilon > 0 \) sufficiently small so that the allocation \( x^\ast \) is "\( \eta \)-envy-free" in the following sense. For any agents \( i \) and \( j \), if \( x_i^\ast \in X \) and \( u_i(x_i^\ast) > u_i(x'_i) \), that is, if \( i \) envies \( j \), then there is a commodity \( l \) such that \( u_i(x_i^\ast) \geq u_i(y_i) \) for every \( y_i \in X \) such that \( y_i \leq x_i^\ast - \eta \epsilon_1 \). In particular, if \( X_\ast = \{0, 1, \ldots, k\}^L \) as in Budish (2011), then we can choose \( y_i = x_i^\ast - \epsilon_1 \) and \( x^\ast \) is 1-envy-free, that is, envy-free up to a single unit of a single good.

**Remark 5 ((I + 1)–maxmin utility).** If \( \epsilon_1 = \cdots = \epsilon_I = 1 \) and \( \epsilon < 1/I \), then \( u_i(x_i^\ast) \geq \max(\min(u_i(y_1), u_i(y_2), \ldots, u_i(y_{I+1})) \) for each agent \( i \), where the maximum is over all \( y_1, \ldots, y_{I+1} \in X \) such that \( \Sigma_{j=1}^{I+1} y_j \leq \omega. \) That is, \( x^\ast \) is at least as good for \( i \) as his \((I + 1)\)-maxmin share (Budish 2011).\(^{13}\)

**Remark 6 (Weak stability and efficiency).** There do not exist \( S \subseteq \{1, \ldots, I\} \) and \( \hat{x} \in X \) such that \( \hat{x} \neq x^\ast \) for at least one \( i \in S \), \( \Sigma_{i\in S} \hat{x}_i \leq \Sigma_{i\in S} x^\ast_i \) for every \( l \) with \( p^l > 0 \), and \( u_i(x^\ast_i) > u_i(y^\ast_i) \) for every \( i \in S \) such that \( \hat{x} \neq x^\ast \).\(^{14}\) In particular, setting \( S = \{1, \ldots, I\} \) shows that \( x^\ast \) is weakly Pareto efficient in this economy for any aggregate endowment vector \( \omega^\ast \) that satisfies \( \omega^\ast \geq \Sigma x^\ast_i \) if \( p^l > 0 \), and \( \omega^\ast \geq \Sigma x^\ast_i \) if \( p^l = 0 \). If, as in Budish (2011), all \( X_i \) are finite and preferences exhibit no indifference, then weak stability and efficiency are equivalent to standard stability and efficiency, wherein blocking requires only one individual in the coalition to be made strictly better off.

The next two remarks indicate that the efficiency and maxmin results described in the previous remarks can be improved on with an arbitrarily small degradation of the bound in part b of theorem 1. Fix any arbitrarily small \( \eta > 0 \). Choose \( \bar{\epsilon} > 0 \) small enough so that \( \bar{\epsilon}I/\sqrt{L} < (\Sigma_{i=1}^{I} p_i) \eta \) for every

\(^{12}\) Given \( \eta > 0 \), choose \( \epsilon > 0 \) so that \( \max p_i > \epsilon/\eta \) for every price vector \( p \in [0, \infty]^L \) such that \( p_i \geq 1 \) for some consumer \( j \) and some \( x_j \in X \). Such an \( \epsilon \) exists by the compactness of the consumption sets. With this choice of \( \epsilon \), suppose that \( u_i(x_i^\ast) > u_i(x'_i) \). Then \( p^l x_i^\ast > 1 \), and so there is \( \hat{x} \) such that \( \hat{p}_i \eta > \epsilon \). Hence, because also \( p^l x_i^\ast \geq 1 + \epsilon \) by part a(i), we have \( p^l(x_i^\ast - \eta \epsilon_1) < 1 \). So any \( y_i \leq x_i^\ast - \eta \epsilon_1 \) that is in \( x^\ast \) satisfies \( p^l y_i < 1 = \epsilon_1 \), and so \( u_i(x_i^\ast) \geq u_i(y_i) \) by part a(ii).

\(^{13}\) Otherwise, if \( y_1, \ldots, y_{I+1} \) is a solution to the maxmin problem, then \( u_i(y_i) > u_i(x_i^\ast) \) for every \( j \). But then by part a(ii), \( p^l y_i > 1 \) for every \( j \), and so \( p^l \Sigma_{j=1}^{I+1} y_j > I + 1 > (1 + \epsilon)I \geq \hat{p}^l \omega \), where the final inequality is by part \( \epsilon \). But the outer strict inequality contradicts the feasibility of \( y_1, \ldots, y_{I+1} \) for the maxmin problem.

\(^{14}\) If such an \( \hat{x} \) were to exist, then part a(ii) implies that \( \hat{p}^l \hat{x}_i > \hat{p}^l x_i^\ast \) for every \( i \in S \) such that \( \hat{x} \neq x_i^\ast \). Since there is at least one \( i \in S \) such that \( \hat{x} \neq x_i^\ast \), \( \hat{p}^l \Sigma_{i\in S} \hat{x}_i > \hat{p}^l \Sigma_{i\in S} x_i^\ast \). Hence, there is an \( l \) such that \( \hat{p}_l > 0 \) and \( \Sigma_{i\in S} \hat{x}_i > \Sigma_{i\in S} x_i^\ast \), contradicting the feasibility of \( \hat{x} \) for the coalition \( S \).
price vector \( p \in [0, \infty)^L \) satisfying \( \max_{i \in X} p \xi_i \geq 1 \). Such an \( \bar{e} > 0 \) exists by the compactness of \( X \).

Remark 7 (Maxmin utility). If \( c_1 = \cdots = c_l = 1 \), if \( e < \bar{e} \), and if the bound in part \( b \) is weakened to \( \| z^* \| \leq \eta + \delta_i(p^*, e)\sqrt{l}/2 \), then we can ensure that \( u_i(x_i^*) \geq \max(u_i(y_1), u_i(y_2), \ldots, u_i(y_l)) \), where the maximum is over all \( y_1, \ldots, y_l \in X \) such that \( \Sigma_{i=1}^l y_i \leq \omega \). That is, \( x^* \) yields agent \( i \) at least his maxmin utility.\(^{15}\)

Remark 8 (Weak Pareto efficiency). If the bound in part \( b \) is weakened to \( \| z^* \| \leq \eta + \delta_i(p^*, e)\sqrt{l}/2 \), then we can ensure that there does not exist \( x \in X \) distinct from \( x^* \) such that \( \Sigma_{i=1}^l x_i \leq \omega \) for every \( l \), and \( u_i(\hat{x}_i) > u_i(x_i^*) \) for every \( i \) such that \( \hat{x}_i \neq x_i^* \).\(^{16}\)

Remark 9. The relevance of the efficiency and fairness results in the previous remarks is called into question by the possibility that \( x^* \) might not be feasible. This difficulty can be mitigated as follows. Define \( \delta_i(e) = \sup \delta_i(p, c) \), where the supremum is over all \( p \in [0, \infty)^L \). If \( \omega \geq \delta_i(e)\sqrt{l}/2 \) for every \( l \), then an application of theorem 1 using the endowment vector \( \omega = \omega - \delta_i(e)\sqrt{l}/2 \) instead of \( \omega \) yields \( \rho^* \) and \( x^* \) satisfying part \( a \) of theorem 1 and, by part \( b \), satisfying \( \Sigma_{i=1}^l x_i \leq \omega \) for every \( l \) and \( \omega - \delta_i(e)\sqrt{l}/2 \leq \Sigma_{i=1}^l x_i \leq \omega \) for every \( l \) with \( \rho^* > 0 \), and, by part \( c \), satisfying \( p^*(\omega - \delta_i(e)\sqrt{l}/2)1 \leq \Sigma_{i=1}^l x_i \leq \Sigma_{i=1}^l x_i \). All of the efficiency and fairness results in the remarks above then go through, but with respect to the economy with aggregate endowment \( \omega \) instead of \( \omega \). In particular, \( x^* \) is feasible, \( \eta \)-envy-free, and weakly stable (because the \( \eta \)-envy-free and weak stability properties do not depend on the aggregate endowment). However, while \( x^* \) is weakly Pareto efficient using only the aggregate endowment \( \omega \), \( x^* \) need not be Pareto efficient for the actual economy with aggregate endowment \( \omega \). A second way to handle infeasibility is to note that, except for at most \( L \) agents, every agent \( i \) can actually receive his bundle \( x_i^* \).\(^{18}\)

The \( L \) exceptional agents can then be assigned any bundles that are fea-

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\(^{15}\) Let \( y_1, \ldots, y_{l+1} \) be a solution to the maxmin problem. Replace \( \omega \) with \( \omega + (\eta/\sqrt{l})1 \) in the statement of theorem 1. Now suppose, by way of contradiction, that \( u_i(y_1) > u_i(x_i^*) \) for every \( i \). Then by part \( a(ii) \), \( \rho^*_{i1} > 1 \) for every \( i \). Therefore, \( \rho^*\Sigma_{i=1}^l y_i \geq \rho^*\omega + (\eta/\sqrt{l})1 - \epsilon l \), where the final inequality is by part \( c \). But \( \rho^*\Sigma_{i=1}^l y_i \geq l \) implies by our choice of \( \epsilon \) that \( \rho^* (\omega + (\eta/\sqrt{l})1) - \epsilon l > 0 \) and so \( \rho^*\Sigma_{i=1}^l y_i \geq \rho^*\omega \), contradicting the feasibility of \( y_1, \ldots, y_l \) for the maxmin problem.

\(^{16}\) Suppose that such an \( \bar{x} \) exists. Replace \( \omega \) with \( \omega + (\eta/\sqrt{l})1 \) in the statement of theorem 1. Then for every \( i \), either \( u_i(\bar{x}_i) > u_i(x_i^*) \) or \( \bar{x}_i = x_i^* \), and so part \( a(ii) \) implies that \( \rho^*\bar{x}_i \geq 1 \) for every \( i \) with strict inequality whenever \( \bar{x}_i = x_i^* \). Therefore, since \( \bar{x} \neq x^* \), \( \rho^*\Sigma_{i=1}^l x_i \geq l \geq \rho^*\omega + (\eta/\sqrt{l})1 - \epsilon l \), where the second inequality is by part \( c \). But our choice of \( \epsilon \) implies that \( \rho^*(\omega + (\eta/\sqrt{l})1) - \epsilon l > 0 \) and so \( \rho^*\Sigma_{i=1}^l x_i \geq \rho^*\omega \), contradicting the feasibility of \( \bar{x} \).

\(^{17}\) The function \( \delta_i(e) \) exists and is finite because \( \delta_i(p, c) \) is bounded above by \( \max \| x - y \| \), where the maximum is over all \( i \) and all \( x, y \in X \).

\(^{18}\) Observe that \( j \leq L \) in the proof below because \( \Sigma_{i=1}^l (\theta_i - 1) \leq L \) and, in the summand, each \( (\theta_i - 1) \geq 1 \). Hence, \( \omega \geq y^* = y_{i1} + \cdots + y_{i1} + x_{i1} + \cdots + x_i^* \) for some \( y_i^* \) in the convex hull of \( x_i, i = 1, \ldots, j \).
sible given the remaining goods. It then follows as in remarks 6 and 7 that all but the \( L \) exceptional agents receive at least their maxmin utility and that no coalition involving only agents outside the exceptional set can trade among themselves following the assignment so that each is better off.

**Remark 10.** For any closed subset \( S \) of \( \mathbb{R}^L \) we may follow Starr (1969) and define

\[
\rho(S) = \sup_{p \in \mathbb{R}^L} \inf_{S \in \mathbb{R}^L} \sup_{a \in A} \| a - a' \|.
\]

The function \( \rho \) is a measure of the convexity of the set \( S \), with \( \rho(S) = 0 \) if and only if \( S \) is convex. Defining \( S(p, \epsilon, \epsilon') = \cup_{\epsilon \in \mathbb{R}^L} D(p, \epsilon + \epsilon') \), the proof shows that the bound in part \( b \) can be reduced to \( \sup, \rho(S(p^*, \epsilon, \epsilon')) \sqrt{L}/2 \).

**Remark 11.** If there is a continuum of each type of agent \( i = 1, 2, \ldots, I \), then no single agent would have an incentive to misreport his utility function because he could not affect the price or the bundle that he receives. Furthermore, the per capita market-clearing error, being bounded above by the bound in part \( b \) divided by the (infinite) number of agents, is zero. Hence, as in Hylland and Zeckhauser (1979) and Budish (2011), the solution here can be the basis for an \( \epsilon \)-incentive-compatible mechanism with small per capita market-clearing error when aggregate endowments are proportional to the large but finite number of agents.

### III. Proof of Theorem 1

Fix any \( \epsilon > 0 \). Because every coordinate of \( \omega \) is strictly positive, we may choose a positive real number \( B \) such that

\[
B > \sum_{i=1}^{I} (\epsilon_i + \epsilon)/\omega_i \quad \text{for every } l = 1, \ldots, L. \tag{1}
\]

For any \( \alpha \in \mathbb{R} \), let \( \alpha^+ = \max(\alpha, 0) \).

Define an \( I + 1 \)-person game between players \( i = 0, 1, \ldots, I \) as follows. Player \( i = 0 \)’s set of pure strategies is \( [0, B]^I \). For \( i \in \{1, \ldots, I\} \), player \( i \)’s set of pure strategies is \( X \). For any \( (p, x) \in [0, B]^I \times X \), the players’ payoffs are as follows:

\[
U_i(p, x) = \epsilon u_i(x_i) - (px_i - \epsilon_i)^+ \quad \text{for } i = 1, \ldots, I
\]

and

\[
U_0(p, x) = p \left( \sum_i x_i - \omega \right).
\]

All of the payoff functions \( U_i : [0, B]^I \times X \to \mathbb{R} \) are continuous.
Let us allow each player $i > 0$ to use a mixed strategy, that is, a probability measure, $m_i$ on $X_i$. For $i > 0$, let $M_i$ denote player $i$'s space of mixed strategies, henceforth simply strategies, and let $M = \times_{i=1}^\infty M_i$. Each $M_i$ is endowed with the weak* topology. Like $[0, B]^L$, each $M_i$ is nonempty, compact, and convex.

For any $m = (m_1, \ldots, m_i) \in M$, let $\bar{m}$ denote the product measure $m_1 \times \cdots \times m_i$.

Extend the players’ payoff functions to $[0, B]^L \times M$, by an expected utility calculation. Then, all of the payoff functions $U_i : [0, B]^L \times M \to \mathbb{R}$ are continuous. Since each player’s payoff function is quasi-concave (linear in fact) in his own strategy for any fixed strategies of the others, the players’ best reply correspondences satisfy all of the hypotheses of Glicksberg’s (1952) fixed-point theorem. Hence, this game possesses a Nash equilibrium $(\bar{p}^*, \bar{m}^*)$.

For each player $i > 0$, equilibrium implies that the support of $m_i^*$ contains only elements $x_i$ of $X_i$ that maximize $U_i(p^*, x_i)$.

Hence, if $x_i$ is in the support of $m_i^*$, then $x_i$ solves

$$\max_{y \in X_i} \left( \epsilon u_i(y) - (p^* y_i - c_i)^+ \right).$$

Since $u_i$ is nonnegative and $0 \in X_i$, the maximum value in (2) is nonnegative. Consequently, since $u_i$ is bounded above by 1, it must be the case that

$$p^* x_i \leq c_i + \epsilon \quad \text{for every } x_i \text{ in the support of } m_i^*.$$  

Furthermore, any solution $x_i$ to the maximization problem (2), and hence any $x_i$ in the support of $m_i^*$, must solve

$$\max_{y \in X_i} u_i(y) \quad \text{subject to } p^* y_i \leq \max(p^* x_i, c_i).$$

To see this claim, let $x_i$ solve (2). Then for every $y_i \in X_i$,

$$\epsilon u_i(x_i) \geq \epsilon u_i(y_i) - (p^* y_i - c_i)^+ + (p^* x_i - c_i)^+$$

$$\geq \epsilon u_i(y_i) \quad \text{if } p^* y_i \leq \max(p^* x_i, c_i),$$

as claimed.

Define

$$y^* = \int_X \sum_i x_i m^*(dx).$$

19 Recall that the support of a probability measure in a separable metric space is the smallest closed subset having probability one.
Then player 0’s equilibrium payoff, \( U_0(p^*, m^*) \), satisfies
\[
U_0(p^*, m^*) = p^*(y^* - \omega) \geq p(y^* - \omega) \quad \forall p \in [0, B]^l.
\]

Consequently, for any \( l \in \{1, \ldots, L\} \), \( y_i^* < \omega_i \Rightarrow p_i^* = 0 \) and \( y_i^* > \omega_i \Rightarrow p_i^* = B \). But then \( y_i^* > \omega_i \) is impossible since \( p_i^* = B \) implies, by (3), that \( y_i^* \leq \epsilon_i + \varepsilon \)/\( B \) for every agent \( i \) and so \( y_i^* \leq \Sigma_i(\epsilon_i + \varepsilon)/B < \omega_i \) by (1). We may conclude that
\[
y_i^* \leq \omega_i \quad \text{for every } l = 1, \ldots, L
\]
and that \( p_i^* = 0 \) for any \( l \) for which the inequality is strict.

For each \( i = 1, \ldots, I \), let \( e_i \) denote the \( i \)th unit vector \( (0, \ldots, 0, 1, 0, \ldots, 0) \). Then
\[
\int_x \left[\frac{1}{I} \Sigma_{i=1}^I (e_i, x_i)\right] m^*(dx) = \frac{1}{I} (1, \ldots, 1, y^*) \in \Delta_I \times \mathbb{R}^l,
\]
where \((e_i, x_i)\) denotes the concatenation of \( e_i \) and \( x_i \), and \( \Delta_I \) denotes the \( I \)–dimensional unit simplex.

The equality in (6) says that \((1, \ldots, 1, y^*)/I \) is in the convex hull of the closed subset \( C \) of \( \Delta_I \times \mathbb{R}^l \) that consists of all points of the form \((e_i, x_i)\), where \( x_i \) is in the support of \( m_i^* \) for each \( i \). By Caratheodory’s theorem (Rockafellar 1970), \((1, \ldots, 1, y^*)/I \) can therefore be written as a convex combination of \( I + L \) or fewer points belonging to \( C \). Thus, for some positive integer \( K \) we may write
\[
\frac{1}{I} (1, \ldots, 1, y^*) = \sum_{i=1}^I \sum_{k=1}^K \lambda_k (e_i, x_i^k),
\]
where the \( \lambda_k \)'s are nonnegative and sum to one, and at most \( I + L \) of the \( \lambda_k \)'s are strictly positive and \( \lambda_k > 0 \) implies that \( x_i^k \) is in the support of \( m_i^* \).

For each \( i = 1, \ldots, I \), let \( S_i = \{ x_i^k : \lambda_k > 0 \} \). Since the first \( I \) coordinates of the vector on the left-hand side of (7) are positive, each \( S_i \) contains at least one element. Reindexing if necessary, let \( S_1, \ldots, S_j \) denote those \( S_i \) that contain two or more elements. So \( S_{j+1}, \ldots, S_I \) are singletons, and since at most \( I + L \) of the \( \lambda_k \)'s are strictly positive, the union of \( S_1, \ldots, S_j \) contains no more than \( L + j \) elements.

For every \( i = 1, \ldots, j \), every \( x_i \) in \( S_i \) is in the support of \( m_i^* \) and therefore satisfies (3) and solves (4). In particular, for any \( x_i \in S_i \) letting \( c' = (p^* x_i - c)^+ \), we have \( c' \in [0, c] \) and \( x_i \in D(p^*, c + c') \). Consequently, the distance between any two points in \( S_i \) is no greater than \( \delta_i(p^*, c) \), where \( c = (c_1, \ldots, c_I) \). Therefore, the distance between any point in \( S_i \) and the simple average of all of the points in \( S_i \) is no greater than \( \delta_i(p^*, c)(\#S_i - 1)/\#S_i \).
The equality in (7) for the first \( I \) coordinates implies that \( \sum_{k=1}^{K} I \lambda_{ak} = 1 \) for each \( i \), and the equality for the last \( L \) coordinates then implies that \( y^* \) is contained in the sum of the convex hulls of the sets \( S_1, \ldots, S_L \). Hence, \( y^* \) is contained in the convex hull of \( S_1 + \cdots + S_L \). Consequently, by the Shapley-Folkman theorem (see Starr 1969) we may select \( x^*_i \in S_i \) for each \( i = 1, \ldots, I \), so that

\[
\left\| y^* - \sum_{i=1}^{I} x^*_i \right\|^2 \leq \sum_{i=1}^{I} \left[ \frac{(\#S_i - 1) \delta_i(p^*, c)}{\#S_i} \right]^2.
\]

Since \( \#S_i \geq 2 \) for every \( i = 1, \ldots, j \), we have \( [(\#S_i - 1)/(\#S_i)]^2 \leq (\#S_i - 1)/4 \) for every \( i = 1, \ldots, j \). Hence,

\[
\left\| y^* - \sum_{i=1}^{I} x^*_i \right\|^2 \leq \sum_{i=1}^{I} \left( \frac{\#S_i - 1}{\#S_i} \delta_i^2(p^*, c) / 4 \right) \leq \delta_i^2(p^*, c)L/4,
\]

where the second inequality follows because the union of the sets \( S_1, \ldots, S_j \) contains no more than \( L + j \) elements, and so \( \sum_{i=1}^{j} (\#S_i - 1) \leq L \). Hence, we may conclude that

\[
\left\| y^* - \sum_{i=1}^{I} x^*_i \right\| \leq \delta_i(p^*, c)\sqrt{L}/2. \tag{8}
\]

For every \( l \), either \( \sum x^*_i \leq \omega_l \) or \( y^*_l \leq \omega_l < \sum x^*_i \), by (5). Consequently, for every \( l = 1, \ldots, L \),

\[
\max \left( \sum x^*_i - \omega_l, 0 \right) \leq \left| y^*_l - \sum x^*_i \right|.
\]

Hence, by (8) and the fact that \( p^*_l > 0 \) implies that \( y^*_l = \omega_l \) (by [5]), we have

\[
\| z^* \| \leq \delta_i(p^*, c)\sqrt{L}/2,
\]

where for each \( l = 1, \ldots, L \),

\[
z^*_l = \begin{cases} 
\sum x^*_i - \omega_l & \text{if } p^*_l > 0 \\
\max \left( \sum x^*_i - \omega_l, 0 \right) & \text{if } p^*_l = 0,
\end{cases}
\]

which establishes part \( b \) of theorem 1.

\(^{20}\) Because the sum of the convex hulls of any finite number of sets is equal to the convex hull of their sum.
Part a is established by noting that for each $i > 0$, $x^*_i \in S_i$ implies that $x^*_i$ is in the support of $m^*_i$, and so parts i and ii follow by (3) and (4).

Finally, part c is established by noting that $p^\epsilon \omega = p^\epsilon y^* \leq \Sigma_i (c_i + \epsilon)$, where the equality follows from (5) and the inequality follows from (3). QED

Remark 12. If one were to use the game in the proof of theorem 1 as the basis for an algorithm to compute the prices and allocations $p^*$, $x^*$, then one would need to ask agents to report their ordinal preferences (e.g., their indifference maps), not their utility functions. The mechanism would then use some canonical procedure to generate a utility representation. The reason is that if the algorithm maximizes $\epsilon u_i(y_i) - (py_i - c_i)^+$ for each agent $i$, then, for any fixed $\epsilon$, any agent who cannot affect the price would have an incentive to scale up his reported utility numbers $u_i(y_i)$.

References