Maximal Revenue with Multiple Goods: Nonmonotonicity and Other Observations*

Sergiu Hart† Philip J. Reny‡

November 21, 2013

Abstract

Consider the problem of maximizing the revenue from selling a number of goods to a single buyer. We show that, unlike the case of one good, when the buyer’s values for the goods increase the seller’s maximal revenue may well decrease. We then identify two circumstances where monotonicity does obtain: when optimal mechanisms are deterministic and symmetric, and when they have submodular prices. Next, through simple and transparent examples, we clarify the need for and the advantage of randomization when maximizing revenue in the multiple-good versus the one-good case. Finally, we consider “seller-favorable” mechanisms, the only ones that matter when

*Previous versions: June 2011, November 2012 (Center for Rationality DP-630), April 2013. See http://www.ma.huji.ac.il/hart/abs/2good-p.html for a presentation that includes some of the results of this paper. Research partially supported by the European Research Council (FP7-249159) and the National Science Foundation (SES-0922535, SES-1227506). The authors thank Bob Aumann, Elchanan Ben-Porath, Eddie Dekel, Alex Gershkov, Ilan Kremer, Vijay Krishna, Noam Nisan, Motty Perry, Eyal Winter, and Asher Wolinsky for useful discussions and suggestions. We also thank the co-editor and the referees, whose perceptive comments, questions, and suggestions led us to sharpen and improve our results.

†Department of Economics, Institute of Mathematics, and Center for the Study of Rationality, The Hebrew University of Jerusalem. E-mail: hart@huji.ac.il Web site: http://www.ma.huji.ac.il/hart

‡Department of Economics, University of Chicago. E-mail: preny@uchicago.edu Web site: http://home.uchicago.edu/~preny
maximizing revenue. They are essential for our positive monotonicity results, and they also circumvent well-known nondifferentiability issues.

1 Introduction

Consider the problem of a seller who wishes to maximize the revenue from selling multiple goods to a single buyer with private information about his value for the goods. In contrast to the one-good case where a complete solution has been known for years,\footnote{See Myerson (1981), who deals also with multiple buyers.} a general solution in the case of multiple goods remains elusive and, except under special circumstances,\footnote{See, e.g., Armstrong (1996), Thanassoulis (2004), Pycia (2006), Manelli and Vincent (2006, 2007, 2012), Pavlov (2011), and Hart and Nissan (2012).} very little is known about even the form of the solution or its properties. The present paper highlights some important differences between the one-good and the multiple-good cases.

In Section 2, we exhibit the surprising phenomenon that the seller’s maximal revenue may well decrease when the buyer’s values for the goods increase.\footnote{What we compare is the maximal revenue from two given distributions, one having higher values than the other (formally, this means first-order stochastic dominance).} This revenue nonmonotonicity can occur only when there is more than one good: revenue is easily shown to be nondecreasing in the buyer’s value when there is only a single good. Thus, the seemingly clear intuition that the seller is able to extract more revenue from buyers whose valuations for the goods are higher turns out to be false in general.\footnote{When the mechanism is held fixed, there are well-known examples in which the seller’s revenue unexpectedly falls; for instance, when the number of bidders increases (Matthews 1984, Menicucci 2009), and when the seller releases more information (Perry and Reny 1999). But in both of these cases, the seller’s maximal revenue cannot fall since the seller can always choose to ignore the additional bidders, and can always choose not to release new information.}

Under what circumstances will the seller’s maximal revenue increase when the buyer’s distribution of valuations increases? In Section 2.3 we identify two such circumstances: one, when the mechanism is deterministic and symmetric, and the other, when payments to the seller can be described by
a submodular pricing function. These positive revenue-monotonicity results are both interesting in their own right and help explain why counterexamples to monotonicity (ours included) cannot be entirely transparent—they must involve randomizations or asymmetries, and their revenue function cannot be submodular.

In Section 3, we present a simple example where randomization is necessary for revenue maximization, and clarify why randomization is needed only when there are multiple goods.

In Section 4, we formally introduce “seller-favorable” mechanisms, where, when the buyer is indifferent, the tie is always broken in favor of the seller. These mechanisms are the only ones that matter when maximizing the seller’s revenue, and also play an important role in the positive monotonicity results of Section 2.3. We characterize the revenue of seller-favorable mechanisms using directional derivatives, which exist everywhere; this has the additional benefit of circumventing nondifferentiability issues that arise from incentive compatibility, which, while ultimately harmless, are often a distracting nuisance within the analysis.

Since the maximal-revenue problem appears significantly less well-behaved when the values of the goods are correlated (cf.\cite{HartNisan2012, HartNisan2013}), it is important also to obtain (inevitably more subtle) examples with independent, and even independent and identically distributed (i.i.d.), values. We do so both for revenue nonmonotonicity and for randomization.

1.1 Preliminaries

The seller possesses \( k \geq 1 \) goods (or “items”), which are worth nothing to him (and there are no costs). The valuation of the goods to the buyer is given by a vector\(^6\) \( x = (x_1, x_2, ..., x_k) \in \mathbb{R}_+^k \), where \( x_i \geq 0 \) is his value for good \( i \). The valuation is assumed to be additive over the goods: the buyer’s

\(^{5}\)It is shown there, for instance, that deterministic mechanisms always ensure at least one half of the maximal revenue in the independent case, but only an arbitrarily small fraction in the general (correlated) case.

\(^{6}\)\( \mathbb{R} \) is the real line, \( \mathbb{R}^k \) the \( k \)-dimensional Euclidean space, and \( \mathbb{R}_+^k = \{ x \in \mathbb{R}^k : x \geq 0 \} \) its nonnegative orthant. We follow the standard assumption that valuations are nonnegative; in Section 4 we deal with arbitrary valuations.
value of a subset $L \subset \{1, 2, ..., k\}$ of goods is $\sum_{i \in L} x_i$. The buyer knows the valuation vector $x$, whereas the seller knows only that $x$ is drawn from a given probability distribution $F$ on $\mathbb{R}^k_+$. We make no further assumptions on $F$. In particular, $F$ may possess atoms and its support may be finite or infinite, and need not be convex or even connected. The seller and the buyer are risk-neutral and have quasilinear utilities.

A (direct) mechanism for selling the $k$ goods is given by a pair of functions $(q, s)$, where $q = (q_1, q_2, ..., q_k) : \mathbb{R}^k_+ \to [0,1]^k$ and $s : \mathbb{R}^k_+ \to \mathbb{R}$. If the buyer reports that his valuation is $x$, then $q_i(x) \in [0,1]$ is the probability that the buyer receives good $i$ (for $i = 1, ..., k$), and $s(x)$ is the payment that the seller receives from the buyer. We call $q$ the outcome function, and $s$ the payment function; the range $M := \{(q(x), s(x)) : x \in \mathbb{R}^k_+\} \subset [0,1]^k \times \mathbb{R}$ of the mechanism is referred to as its menu. When the buyer reports his valuation $x$ truthfully, his payoff is $b(x) = \sum_{i=1}^k q_i(x)x_i - s(x) = q(x) \cdot x - s(x)$, and the seller’s payoff is $s(x)$. A mechanism $(q, s)$ is individually rational (IR) if $b(x) \geq 0$ for all $x \in \mathbb{R}^k_+$, and it is incentive compatible (IC) if $b(x) \geq q(y) \cdot x - s(y)$ for all $x, y \in \mathbb{R}^k_+$. By the Revelation Principle, the maximal revenue from the distribution $F$ is $\text{Rev}(F) := \sup \mathbb{E}_F [s(x)]$, where $x$ is distributed according to $F$, and the supremum is over all IC and IR mechanisms $(q, s)$.

2 Nonmonotonicity: Increasing Values May Decrease Revenue

When the buyer’s values for the goods increase, what happens to the seller’s maximal revenue? It stands to reason that the revenue should also increase, as there is now more value for the seller to “extract.” While this is easily
shown to be true when there is one good (Section 2.1), it is perhaps a surprise that it no longer holds when there are multiple goods (Sections 2.2 and 2.4).

To see this, consider two situations: in the first the valuation is given by the \((\mathbb{R}_+^k\text{-valued})\) random variable \(X_1\) with distribution function \(F_1\); and in the second, by the random variable \(X_2\) with distribution function \(F_2\). Assume that \(X_1 \leq X_2\) everywhere; i.e., in every state \(\omega\) the realization \(X_1(\omega)\) of \(X_1\) is less than or equal to (in all \(k\) coordinates) the realization \(X_2(\omega)\) of \(X_2\). What we will show is that the maximal revenue that is obtained from \(X_2\) may well be smaller than the maximal revenue that is obtained from \(X_1\). In terms of distributions, the condition \(X_1 \leq X_2\) amounts to first-order stochastic domination of \(F_1\) by \(F_2\).

### 2.1 Monotonicity for one good

When there is only one good, i.e., \(k = 1\), incentive compatibility (IC) implies that a buyer with a higher valuation pays no less than a buyer with a lower valuation. Thus, increasing the valuation of the buyer can only increase the revenue.

**Proposition 1** Let \(F_1\) and \(F_2\) be two distributions on \(\mathbb{R}_+\). If \(F_2\) first-order stochastically dominates \(F_1\) then \(\text{REV}(F_1) \leq \text{REV}(F_2)\).

**Proof.** First, we claim that every IC mechanism is monotonic in the sense that the seller’s payoff increases weakly with the buyer’s value: if \(x > y \geq 0\) then \(s(x) \geq s(y)\). Indeed, for all \(x, y\), the IC inequalities at \(x\) and at \(y\) imply \((q(x) - q(y))x \geq s(x) - s(y) \geq (q(x) - q(y))y\); hence \((q(x) - q(y))(x - y) \geq 0\); when \(x > y\) it follows that \(q(x) - q(y) \geq 0\) and thus \(s(x) - s(y) \geq 0\) (because \(y \geq 0\)).

---

\(^{11}\)Thus, in applications where the distribution of \(X_2\) is not precisely known, and only a certain lower bound \(X_1\) is given, the optimal revenue from \(X_1\) does not necessarily provide a lower bound for the optimal revenue from \(X_2\).

\(^{12}\)Formally, \(F_2\) first-order stochastically dominates \(F_1\) if and only if \(\mathbb{E}_{F_1}[u(X)] \leq \mathbb{E}_{F_2}[u(X)]\) for every nondecreasing function \(u: \mathbb{R}^k \rightarrow \mathbb{R}\). As is well known, this is equivalent to having two random variables \(X_1\) and \(X_2\) with distributions \(F_1\) and \(F_2\), respectively, that are defined on the same probability space and satisfy \(X_1 \leq X_2\) pointwise (this is called “coupling”). A comprehensive treatment of stochastic dominance can be found in Shaked and Shanthikumar (2010).
Second, first-order stochastic dominance implies that $E_{F_1}[s(x)] \leq E_{F_2}[s(x)]$ for every IC mechanism, since $s$ is a nondecreasing function. ■

**Remark.** Proposition 1 also follows easily from Myerson’s (1981) characterization of the optimal revenue when there is one good as $\text{Rev}(F) = \sup_{p \geq 0} p \cdot (1 - F(p))$. However, the proof above shows that revenue monotonicity holds not only for optimal mechanisms, but for *any* incentive-compatible mechanism.

### 2.2 Nonmonotonicity for multiple goods

Surprisingly, the above does not hold when there is more than one good. That is, increasing the buyer’s valuations need *not* yield higher revenue to the seller.

When there are multiple goods one can easily construct examples of IR and IC mechanisms that are not monotonic.\(^{13}\) Take for instance the mechanism where the buyer is offered a choice from the following menu of four outcomes: get nothing and pay nothing (with payoff = 0); or get good 1 for price 1 (with payoff = $x_1 - 1$); or get good 2 for price 2 (with payoff = $x_2 - 2$); or get both goods for price 4 (with payoff = $x_1 + x_2 - 4$); thus, the buyer’s payoff is

$$b(x_1, x_2) = \max\{0, x_1 - 1, x_2 - 2, x_1 + x_2 - 4\}. \quad (1)$$

See Figure 1 for the regions in the buyer’s valuation space where each outcome is chosen. If the valuation of the buyer is, say, $x = (1, 7/3)$, then his optimal choice is to pay 2 for good 2 (so $q(x) = (0, 1)$ and $s(x) = 2$), whereas if his valuation increases to $x' = (2, 7/3)$ (where the first good is worth more), or even to $x'' = (2, 8/3)$ (where both goods are worth more), then his optimal choice is to pay 1 for good 1 (so $q(x') = q(x'') = (1, 0)$ and $s(x') = s(x'') = 1$).

Thus the seller receives a lower payment (1 instead of 2) when the buyer’s values increase. What is happening is that when the value of good 1 goes up (from $x_1 = 1$ to $x'_1 = 2$), the probability of getting good 1 also goes up (i.e., $q_1(x') = 1 > 0 = q_1(x)$; this is a consequence of the convexity of the buyer’s

\(^{13}\)Our first such example was constructed together with Noam Nisan.
payoff function $b$, since $x_2 = x'_2$). However, at the same time the probability of getting good 2 may go down (here, $q_2(x') = 0 < 1 = q_2(x)$), and in such a way, moreover, that the allocation is worth less to the buyer (i.e., $q(x') \cdot x' = 2 < 7/3 = q(x) \cdot x'$) — which, by incentive compatibility, implies that the payment to the seller goes down (since $s(x') - s(x) \leq q(x') \cdot x' - q(x) \cdot x' < 0$).

The difficult question is whether this nonmonotonicity can also occur for the maximal revenue. We provide two examples: a simpler one (below), where the unique optimal mechanism is exactly the mechanism described above,\footnote{This explains the reason for including the outcome $x_1 + x_2 - 4$ in the mechanism.} and a more complicated one (Section 2.4), where the valuations of the two goods are independent and identically distributed.

**Example E1.** For every $0 \leq \alpha \leq 1/4$, let $F_\alpha$ be the following distribution...
on $\mathbb{R}^2$: 

$$
\mathcal{F}_\alpha = \begin{cases} 
(1, 1), & \text{with probability } 1/4, \\
(1, 2), & \text{with probability } 1/4 - \alpha, \\
(2, 2), & \text{with probability } \alpha, \\
(2, 3), & \text{with probability } 1/2.
\end{cases}
$$

As $\alpha$ increases, probability mass is moved from $(1, 2)$ to $(2, 2)$, and so $\mathcal{F}_\alpha$ first-order stochastically dominates $\mathcal{F}_{\alpha'}$ when $\alpha > \alpha'$. Nevertheless, the maximal revenue $\text{Rev}(\mathcal{F}_\alpha)$ decreases with $\alpha$ (in the region $0 \leq \alpha \leq 1/12$).

**Proposition 2** In Example E1: for every $0 \leq \alpha \leq 1/12$,

$$
\text{Rev}(\mathcal{F}_\alpha) = 11/4 - \alpha.
$$

**Proof.** First, revenue of $11/4 - \alpha$ is achieved by the mechanism given in the following table (whose buyer payoff function is (1)).

<table>
<thead>
<tr>
<th>Valuation $x = (x_1, x_2)$</th>
<th>Outcome $q(x) = (q_1(x), q_2(x))$</th>
<th>$s(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 1)$</td>
<td>$(1, 0)$</td>
<td>1</td>
</tr>
<tr>
<td>$(2, 2)$</td>
<td>$(1, 0)$</td>
<td>1</td>
</tr>
<tr>
<td>$(1, 2)$</td>
<td>$(0, 1)$</td>
<td>2</td>
</tr>
<tr>
<td>$(2, 3)$</td>
<td>$(1, 1)$</td>
<td>4</td>
</tr>
</tbody>
</table>

Indeed, revenue is $(1/4) \cdot 1 + (1/4 - \alpha) \cdot 2 + \alpha \cdot 1 + (1/2) \cdot 4 = 11/4 - \alpha$.

Second, we show that a higher revenue cannot be obtained. Consider the following inequalities:

$$
\begin{align*}
q_{11} + q_{12} - s_{11} &\geq 0 \\
q_{12} + 2q_{12} - s_{12} &\geq q_{11} + 2q_{12} - s_{11} \\
2q_{12} + 2q_{22} - s_{22} &\geq 2q_{11} + 2q_{12} - s_{11} \\
2q_{12} + 3q_{23} - s_{23} &\geq 2q_{11} + 3q_{12} - s_{11} \\
2q_{12} + 3q_{23} - s_{23} &\geq 2q_{11} + 3q_{22} - s_{22} \quad | \\
\end{align*}
$$

(3)
(the first inequality is IR at (1, 1), and the others are various IC constraints). Multiplying these inequalities by the multipliers on the right (which are all nonnegative when $0 \leq \alpha \leq 1/12$) and then adding them up yields:

$$- (3/4 - 3\alpha) q_{11}^2 - 2\alpha q_{12}^1 + (1/4 - 3\alpha) q_{11}^2 + 2\alpha q_{12}^2 + q_{13}^3 + (3/2)q_{23}^2$$

$$\geq (1/4)s_{11}^2 + (1/4 - \alpha) s_{12}^2 + \alpha s_{22}^2 + (1/2)s_{23}^2.$$ 

The right-hand side is precisely the expected revenue at $F_\alpha$, and the left-hand side is at most $0 + 0 + (1/4 - 3\alpha) + 2\alpha + 1 + 3/2 = 11/4 - \alpha$ (since $q_{21}^1, q_{12}^2 \geq 0$ and $q_{22}^1, q_{23}^2, q_{23}^3 \leq 1$). Therefore the revenue cannot exceed $11/4 - \alpha$, and so the revenue of $11/4 - \alpha$ achieved by (2) is indeed maximal.

Remarks. (a) The mechanism (2) is the unique optimal mechanism at each $F_\alpha$ with $0 \leq \alpha < 1/12$; indeed, in order to get revenue $11/4 - \alpha$ one needs all relevant inequalities to become equalities (thus $q_{21}^1 = q_{12}^2 = 0$ and $q_{22}^1 = q_{23}^2 = q_{23}^3 = 1$, which together with (3) as equalities can be easily shown to yield $q_{11}^1 = 1, q_{22}^1 = 0, s_{11}^1 = 1, s_{12}^1 = 2, s_{22}^2 = 1, s_{23}^3 = 4$—which is precisely (2)).

(b) Any small enough perturbation of the example—such as having full support on, say $[0, 4]^2$, or strictly increasing all valuations as $\alpha$ increases—will not affect the nonmonotonicity, since the inequality $\text{REV}(F_0) > \text{REV}(F_{1/12})$ is strict.

(c) There are special cases where increasing values yields monotonicity of the maximal revenue. When the valuations are increased uniformly—i.e., each $x \in \mathbb{R}_+^k$ is replaced by $x + d$ for fixed $d \in \mathbb{R}_+^k$—then the optimal revenue cannot decrease. Indeed, for every mechanism $(q, s)$, let $(\tilde{q}, \tilde{s})$ be the mechanism given by $\tilde{q}(x + d) := q(x)$ and $\tilde{s}(x + d) := s(x) + q(x) \cdot d$; it is immediate to check that if $(q, s)$ is IC then so is $(\tilde{q}, \tilde{s})$. Let $X$ be an $\mathbb{R}_+^k$-valued random variable; then\(^{15}\) $\text{REV}(X + d) \geq \mathbb{E}[\tilde{s}(X + d)] = \mathbb{E}[s(X)] + \mathbb{E}[q(X)] \cdot d \geq \mathbb{E}[s(X)]$; this holds for every IC mechanism $(q, s)$, and so\(^{15}\)

\(^{15}\)We slightly abuse the notation and write $\text{REV}(X)$ instead of $\text{REV}(F)$ when $X$ is a random variable with distribution $F$. 

9
\( \text{REV}(X + d) \geq \text{REV}(X) \).

When the valuations are proportionately increased uniformly—i.e., each \( x \in \mathbb{R}^k_+ \) is replaced by \( c \ast x = (c_1 x_1, c_2 x_2, \ldots, c_k x_k) \) for fixed \( c_1, c_2, \ldots, c_k \geq 1 \)—then, again, the optimal revenue cannot decrease. To see this, for every mechanism \((q, s)\), let \((\hat{q}, \hat{s})\) be given by \( \hat{q}_i(c \ast x) := q_i(x)/c_i \) for \( i = 1, \ldots, k \), and \( s(c \ast x) := s(x) \); if \((q, s)\) is IC then so is \((\hat{q}, \hat{s})\) (since \( \hat{q}(c \ast y) \cdot (c \ast x) - s(c \ast y) = q(y) \cdot x - s(y) \) for every \( x, y \in \mathbb{R}^k_+ \)). Therefore \( \text{REV}(c \ast X) \geq \mathbb{E}[\hat{s}(c \ast X)] = \mathbb{E}[s(X)] \), and so
\[
\text{REV}(c \ast X) \geq \text{REV}(X).
\]

To conclude this section, observe that Example E1, whose optimal mechanism is deterministic, shows that nonmonotonicity for multiple goods is not due to the need for lotteries in this case (see Section 3).

### 2.3 Some classes of monotonic mechanisms

In this section, we identify two natural classes of mechanisms for which monotonicity is guaranteed to hold in the multiple-good case: mechanisms that are symmetric and deterministic, and mechanisms whose pricing is submodular.\(^{17}\)

Recall Example E1. The optimal mechanism there, which is deterministic, is not symmetric across the goods, since the price of good 1 is different from the price of good 2. Also, it is not subadditive (or submodular), since the price of the bundle is larger than the sum of the prices of the separate goods. The results of this section will imply that any counterexample to nonmonotonicity must satisfy these conditions.

\(^{16}\)One cannot get the opposite inequality by interchanging \( X \) and \( c \ast X \), since \( c_i \geq 1 \) is needed in order to guarantee that \( \hat{q}_i \leq 1 \). However, if all the \( c_i \) are equal, say \( c_i = c \), then taking \( \hat{q}(\gamma x) := q(x) \) and \( \hat{s}(\gamma x) := s(x) \) (instead of \((\hat{q}, \hat{s})\)) yields \( \text{REV}(\gamma X) = \gamma \text{REV}(X) \), for every \( \gamma \geq 0 \). Combining all these we get

\[
\left( \min_i c_i \right) \text{REV}(X) \leq \text{REV}(c \ast X + d) \leq \left( \max_i c_i \right) \text{REV}(X) + \sum_{i=1}^k d_i,
\]

for every fixed \( c_i, d_i \geq 0 \) (use \( c_i = c'_i \gamma \) where \( \gamma = \min_i c_i \) and \( c'_i \geq 1 \)). We thank a referee, whose comment led us to consider these affine transformations.

\(^{17}\)Mechanisms in one or the other of these two classes arise as optimal mechanisms in various examples within the literature.
A mechanism \((q, s)\) is deterministic if the range \(Q := \{q(x) : x \in \mathbb{R}_+^k\}\) of the outcome function \(q\) is included in \([0, 1]^k\) (rather than \([0, 1]^k\); recall Section 1.1). Incentive compatibility implies that to each outcome that appears in the mechanism, i.e., each \(g \in Q\), there is a unique payment: if \(q(x) = q(y)\) then \(s(x) = s(y)\). Thus the menu \(M = \{(q(x), s(x)) : x \in \mathbb{R}_+^k\}\) of a deterministic mechanism is finite (there are at most \(2^k\) outcomes), and such a mechanism can be represented by specifying the payment—or “price”—\(p_I\) for each subset \(I\) of goods, i.e., for each \(I \subset K := \{1, 2, \ldots, k\}\): thus, \(b(x) = \max_{I \subset K} \{\sum_{i \in I} x_i - p_I\}\). A deterministic mechanism is symmetric if \(p_I\) depends only on the size \(|I|\) of \(I\), i.e., the price depends only on the number of goods (and so the function \(b\) is symmetric, i.e., invariant under permutations of the coordinates \(x_1, \ldots, x_k\)).

**Proposition 4** Let \((q, s)\) be a deterministic and symmetric seller-favorable IC mechanism on \(\mathbb{R}_+^k\). Then the payment function \(s\) is nondecreasing.

\(^{18}\)If \(I\) is not a possible outcome of the mechanism, put \(p_I = \infty\).
Proof. With a slight abuse of notation we will write $p_i$ instead of $p_I$ for all $I \subset K$ with $|I| = i$. For $x \in \mathbb{R}_+^k$, let $x^\#$ denote the vector obtained by permuting the coordinates of $x$ so that they are in nonincreasing order, i.e., $x_1^\# \geq x_2^\# \geq \ldots \geq x_k^\#$; symmetry implies that $s(x) = s(x^\#)$. Let $y, z \in \mathbb{R}_+^k$ be such that $y \leq z$, then it easily follows that $y^\# \leq z^\#$, and so we need to show that $s(y^\#) \leq s(z^\#)$. Put $\eta_i := y_1^\# + \ldots + y_i^\# - p_i$ and $\zeta_i := z_1^\# + \ldots + z_i^\# - p_i$; then $b(y^\#) = \max_{1 \leq i \leq k} \eta_i$ and $b(z^\#) = \max_{1 \leq i \leq k} \zeta_i$ (since, for the same price, the buyer always prefers the highest-valued goods). Let $s(y^\#) = p_j$ and $s(z^\#) = p\ell$; then these maxima are attained at $j$ and $\ell$, respectively, and, in particular, we have $\eta_j \geq \eta_\ell$ and $\zeta_\ell \geq \zeta_j$. If $\ell \geq j$ then $p\ell - p_j = (\eta_j - \eta_\ell) + (y_{j+1}^\# + \ldots + y_\ell^\#) \geq 0$ (since $y^\# \geq 0$), and so $p\ell \geq p_j$. If $\ell < j$ then $\zeta_j - \zeta_\ell = (\eta_j - \eta_\ell) + (z_{\ell+1}^\# - y_{\ell+1}^\#) + \ldots + (z_j^\# - y_j^\#) \geq 0$ (since $z^\# \geq y^\#$); but $\zeta_\ell \geq \zeta_j$, and so $\zeta_j = \zeta_\ell$. Therefore $\zeta_j$ is also a maximizer for $b(z^\#)$; since $s(z^\#) = p\ell$, the seller-favorable condition implies that $p\ell \geq p_j$, which completes the proof.

Corollary 5 Let $\text{DSRev}$ denote the optimal revenue obtained when restricted to deterministic and symmetric mechanisms. If $\mathcal{F}_2$ first-order stochastically dominates $\mathcal{F}_1$ then $\text{DSRev}(\mathcal{F}_1) \leq \text{DSRev}(\mathcal{F}_2)$.

Proof. Let $(q, s)$ be a deterministic and symmetric mechanism, and let $M$ be its menu, which is finite. Apply Proposition 15 in Section 4 to get a seller-favorable IC mechanism $(\tilde{q}, \tilde{s})$ that is symmetric (use the construction in the proof there for $x$ with $x_1 \geq x_2 \geq \ldots \geq x_k$, and then extend to all $x$ by $(\tilde{q}(x), \tilde{s}(x)) = (\tilde{q}(x^\#), \tilde{s}(x^\#))$; that is, break ties so that symmetry is preserved). Since the menu $M$ of $(q, s)$ satisfies $\tilde{M} \subset \text{cl} M = M$ (as $M$ is a finite set), the new mechanism is also deterministic. Thus $(\tilde{q}, \tilde{s})$ is deterministic and symmetric. Because $\tilde{s}(x) \geq s(x)$ for all $x$, it follows that $\text{DSRev}$ is in fact the optimal revenue over deterministic, symmetric, and seller-favorable mechanisms. Proposition 4 says that the payment function $\tilde{s}$ of such a mechanism is monotonic, which implies that $\mathbb{E}_{\mathcal{F}_1} [\tilde{s}(x)] \leq \mathbb{E}_{\mathcal{F}_2} [\tilde{s}(x)]$ by stochastic dominance, and hence our result.
Proof of Theorem 3. We have

\[ \text{Rev}(\mathcal{F}_1) = \text{DSRev}(\mathcal{F}_1) \leq \text{DSRev}(\mathcal{F}_2) \leq \text{Rev}(\mathcal{F}_2) \]

(the first equality by assumption, the second by Corollary 5, and the third since DSRev considers only a subclass of mechanisms). □

Remark. Each of the three conditions in Proposition 4—deterministic, symmetric, and seller-favorable—is indispensable: dropping any one of them allows mechanisms with nonmonotonic \( s \). Indeed: Example E2 in Section 2.4 below is not deterministic; Example E1 in Section 2.2 is not symmetric; and the mechanism with buyer payoff function \( b(x_1, x_2) = \max\{0, x_1 - 2, x_2 - 2, x_1 + x_2 - 3\} \) and \( q(2, 1) = (1, 1), s(2, 1) = 3, q(2, 2) = (1, 0), \) and \( s(2, 2) = 2, \) is not seller-favorable.\(^{19}\)

A deterministic mechanism is submodular (Manelli and Vincent 2006) if the payments satisfy \( p_I + p_J \geq p_{I \cup J} + p_{I \cap J} \) for all subsets of goods \( I, J \subset K := \{1, 2, ..., k\} \) (this is a generalization of subadditivity, which obtains when \( I \) and \( J \) are disjoint); alternatively, \( p_{I \cup L} - p_I \) is monotonically decreasing in \( I \subset K \setminus L \), which is a standard decreasing marginal price property. We now generalize this condition to arbitrary (i.e., nondeterministic) mechanisms.

Let \((q, s)\) be an IC mechanism with buyer payoff function \( b : \mathbb{R}_+^k \to \mathbb{R} \). A pricing function \( p : [0, 1]^k \to \mathbb{R} \cup \{\infty\} \) associates a price \( p(g) \) to every lottery \( g \in [0, 1]^k \) (we allow infinite prices for lotteries that are not used). Given a pricing function \( p \), consider offering to the buyer the choices \((g, p(g))\) for all possible lotteries \( g \in [0, 1]^k \); the resulting payoff to the buyer is \( b_p(x) := \sup_{g \in [0,1]^k} (g \cdot x - p(g)) \). We say that \( p \) is a pricing function for the mechanism \((q, s)\) if it yields the same choices and payoffs, i.e., \( p(q(x)) = s(x) \) and \( b(x) = b_p(x) \) for every \( x \in \mathbb{R}_+^k \). Thus,

\[ b(x) = q(x) \cdot x - s(x) = q(x) \cdot x - p(q(x)) = \sup_{g \in [0,1]^k} (g \cdot x - p(g)) \quad (4) \]

\(^{19}\)This example, whose payments are subadditive, shows that seller favorability is indispensable also for the result of Proposition 7 below.
for every $x \in \mathbb{R}^k_+$. While the price of each lottery $q(x)$ that is used in $(q, s)$ is determined,\(^{20}\) there is freedom in choosing the prices of the other lotteries. However, as we will show in Section 4.1, there is a canonical pricing function for $(q, s)$, namely, the lowest one possible; it is given by the function $p_0$, defined as\(^{21}\) $p_0(g) := \sup_{x \in \mathbb{R}^k_+} (g \cdot x - b(x))$.

We say that the mechanism $(q, s)$ is submodular if it has a pricing function $p$ that satisfies
\[
p(g) + p(g') \geq p(g \lor g') + p(g \land g')
\]
for all $g$ and $g'$ in the range, i.e., $g, g' \in Q := q(\mathbb{R}^k_+)$, where $g \lor g'$ and $g \land g'$ denote, respectively, the coordinatewise maximum and minimum of the two vectors $g$ and $g'$. In Section 4.1 we will show that this is equivalent to requiring the canonical pricing function $p_0$ to satisfy (5).

**Theorem 6** Let $\mathcal{F}_1$ be a distribution on $\mathbb{R}^k_+$ such that there is a submodular IC mechanism that is optimal for $\mathcal{F}_1$. Then $\text{REV}(\mathcal{F}_1) \leq \text{REV}(\mathcal{F}_2)$ for every distribution $\mathcal{F}_2$ on $\mathbb{R}^k_+$ that first-order dominates $\mathcal{F}_1$.

The proof will use the following Proposition.

**Proposition 7** Let $(q, s)$ be a submodular seller-favorable IC mechanism. Then the payment function $s$ is nondecreasing.

**Proof.** Let $p$ be a pricing function for $(q, s)$ (thus (4) holds for every $x \in \mathbb{R}^k_+$) that satisfies (5) for every $g, g' \in Q = q(\mathbb{R}^k_+)$. Let $y, z \in \mathbb{R}^k_+$ be such that $y \leq z$; we will show that $s(y) \leq s(z)$.

Put $g_y := q(y)$ and $g_z := q(z)$; then $s(y) = p(g_y)$ and $s(z) = p(g_z)$. From (4) for $y$ we get in particular
\[
g_y \cdot y - p(g_y) \geq (g_y \land g_z) \cdot y - p(g_y \land g_z),
\]
and from (4) for $z$ we get in particular
\[
g_z \cdot z - p(g_z) \geq (g_y \lor g_z) \cdot z - p(g_y \lor g_z).
\]

\(^{20}\)The graph of $p$ restricted to $Q := q(\mathbb{R}^k_+)$ is precisely the menu $M = (q, s)(\mathbb{R}^k_+)$.\(^{21}\) $p_0$ is thus the Fenchel conjugate of the function $b$ (with the usual convention that $b(x) = \infty$ for $x$ outside the domain, i.e., $x \notin \mathbb{R}^k_+$; or, as in Section 4, extend $b$ to all $\mathbb{R}^k$).
Subtracting \( g_z \cdot (z - y) \) from the left-hand side of (7), and the larger \((g_y \lor g_z) \cdot (z - y)\) from its right-hand side (it is larger since \(g_y \lor g_z \geq g_z\) and \(z - y \geq 0\)), gives
\[
g_z \cdot y - p(g_z) \geq (g_y \lor g_z) \cdot y - p(g_y \lor g_z). \tag{8}
\]
Adding (6) and (8) and using \(g_y + g_z = g_y \lor g_z + g_y \land g_z\) (this equality holds for each coordinate) implies
\[
0 \geq p(g_y) + p(g_z) - p(g_y \lor g_z) - p(g_y \land g_z),
\]
which must therefore be an equality by the submodularity condition (5). Therefore we must have equalities throughout, in particular in (7):
\[
b(z) = g_z \cdot z - p(g_z) = (g_y \lor g_z) \cdot z - p(g_y \lor g_z). \tag{9}
\]
Hence the maximum in \(b(z)\) is attained also at \(g = g_y \lor g_z\). Hence, the mechanism \((q', s')\) that is identical to \((q, s)\) except at \(z\), where we put \((q(z'), s'(z)) := (g_y \lor g_z, p(g_y \lor g_z))\), is IC and has the same buyer payoff function \(b\) (by (4) and (9)). Since \((q, s)\) is seller-favorable it follows that \(s'(z) \leq s(z)\), or
\[
p(g_y \lor g_z) \leq p(g_z). \tag{10}
\]
Finally, \(g_y \cdot y - p(g_y) \geq (g_y \lor g_z) \cdot y - p(g_y \lor g_z) \geq g_y \cdot y - p(g_y \lor g_z)\) (the first inequality from (4) for \(y\), and the second since \(g_y \lor g_z \geq g_y\) and \(y \geq 0\)); hence
\[
p(g_y \lor g_z) \geq p(g_y). \tag{11}
\]
Combining (10) and (11) yields \(s(y) = p(g_y) \leq p(g_z) = s(z)\). \(\blacksquare\)

**Corollary 8** Let \(\text{SbRev}\) denote the optimal revenue obtained when restricted to submodular mechanisms. If \(\mathcal{F}_2\) first-order stochastically dominates \(\mathcal{F}_1\) then \(\text{SbRev}(\mathcal{F}_1) \leq \text{SbRev}(\mathcal{F}_2)\).

**Proof.** Given any submodular IC mechanism \((q, s)\), Corollary 19 (with \(D = \mathbb{R}_+^k\)) yields a submodular seller-favorable IC mechanism \((\tilde{q}, \tilde{s})\) with \(\tilde{s}(x) \geq s(x)\) for every \(x \in \mathbb{R}_+^k\). Therefore \(\text{SbRev}\) is in fact the optimal revenue over
submodular seller-favorable mechanisms. Since the payment function $\tilde{s}$ for such mechanisms is monotonic by applying Proposition 7 to $(\tilde{q}, \tilde{s})$, the result follows.

\textbf{Proof of Theorem 6.} Similar to the proof of Theorem 3, using Corollary 8:

$$\text{Rev}(\mathcal{F}_1) = \text{SbRev}(\mathcal{F}_1) \leq \text{SbRev}(\mathcal{F}_2) \leq \text{Rev}(\mathcal{F}_2).$$

\textbf{Remark.} The proof of Proposition 7 shows that $p(g_z) \geq p(g_y \lor g_z)$ and $b(z) = g_z \cdot z - p(g_z) = (g_y \lor g_z) \cdot z - p(g_y \lor g_z)$ (recall that $g_y = q(y)$ and $g_z = q(z)$), from which it follows that $(g_y \lor g_z) \cdot z \leq g_z \cdot z$. Therefore we have equality (since $g_y \lor g_z \geq g_z$ and $z \geq 0$), which implies that $(g_y \lor g_z)_i = (g_z)_i$, i.e. $q_i(y) \leq q_i(z)$, for every coordinate $i$ such that $z_i > 0$. Thus $q$ is coordinatewise nondecreasing on $\mathbb{R}^k_+$. Moreover, $q$ can be adjusted on the boundary of $\mathbb{R}^k_+$ without changing $b$ or $s$, while maintaining IC, and such that the redefined $q$ is coordinatewise nondecreasing on all of $\mathbb{R}^k_+$.

\subsection{2.4 Nonmonotonicity for independent and identically distributed goods}

We now provide an example of nonmonotonicity where two goods are independent and identically distributed, and we increase the values of both goods; that is, we compare the optimal revenue of $F_1 \times F_1$ with that of $F_2 \times F_2$, where $F_1$ and $F_2$ are one-dimensional distributions such that $F_2$ first-order stochastically dominates $F_1$.

The example must be more complicated than Example E1, for two main reasons. First, its optimal mechanisms cannot be submodular (by Theorem

\footnote{Redefine $q(y)$ for $y$ on the boundary of $\mathbb{R}^k_+$ by making its $i$th coordinate zero if $y_i = 0$. Hence, for $y \leq z$, if $z_i = 0$ then $y_i = 0$ and so $q_i(y) = 0 \leq q_i(z)$. Therefore $q_i(y) \leq q_i(z)$ holds now for all coordinates $i$ when $y, z \in \mathbb{R}^k_+$ are such that $y \leq z$. Since the probabilities $q_i(y)$ have only gone down and $q(y) \cdot y$ has not been affected, the redefined mechanism $(q, s)$ remains IC on $\mathbb{R}^k_+$, and the buyer’s payoff function is unchanged.}
and, if it is unique,\(^\text{23}\) it cannot be deterministic (since an i.i.d. distribution is symmetric and thus there always exists a symmetric optimal mechanism, to which we apply Theorem 3). Second, because the two distributions are i.i.d., it is not enough to find an optimal mechanism whose payment function \(s\) is nonmonotonic at a single point, since, by independence, a change in a value of one good applies to all values of the other good. To understand this last point, assume that we increase a value \(\bar{x}_1\) of good 1 to \(\bar{x}_1 + \delta\) for some \(\delta > 0\); the \((\bar{x}_1, x_2)\) valuations now become \((\bar{x}_1 + \delta, x_2)\), for all \(x_2\). While \(s\) may be nonmonotonic at some \((\bar{x}_1, x_2)\), it need not be nonmonotonic at all \((\bar{x}_1, x_2)\); what is needed is that the expectation of \(s(\bar{x}_1, x_2)\) over \(x_2\) be nonmonotonic in \(\bar{x}_1\).

**Example E2.** Let \(F_1\) and \(F_2\) be the following one-dimensional distributions:

\[
F_1 = \begin{cases} 
10, & \text{with probability } \frac{4}{15}, \\
46, & \text{with probability } \frac{1}{90}, \\
47, & \text{with probability } \frac{1}{90}, \\
80, & \text{with probability } \frac{7}{30}, \\
100, & \text{with probability } \frac{7}{45}.
\end{cases}
\]

\[
F_2 = \begin{cases} 
10, & \text{with probability } \frac{2399}{9000}, \\
13, & \text{with probability } \frac{1}{9000}, \\
46, & \text{with probability } \frac{1}{90}, \\
47, & \text{with probability } \frac{1}{3}, \\
80, & \text{with probability } \frac{7}{30}, \\
100, & \text{with probability } \frac{7}{45}.
\end{cases}
\]

Clearly \(F_2\) first-order stochastically dominates \(F_1\) (since \(F_2\) is obtained from \(F_1\) by moving a probability mass of \(1/9000\) from \(10\) to \(13\)), which of course implies that \(F_2 \times F_2\) first-order stochastically dominates \(F_1 \times F_1\). However, the optimal revenue from \(F_1 \times F_1\) turns out to be higher than the optimal revenue from \(F_2 \times F_2\).

**Proposition 9** In Example E2: \(F_2\) first-order stochastically dominates \(F_1\), and

\[
\text{Rev}(F_2 \times F_2) < \text{Rev}(F_1 \times F_1).
\]

\(^{23}\)Which, for discrete distributions, is generically true (i.e., for small perturbations of the distribution, the optimal mechanism is unique).

\(^{24}\)This “accounting” is not entirely correct, as we also need to consider the valuations \((x_2, x_0^1)\) for all \(x_2\), and then \((x_1^0, x_1)\) is counted twice.
**Proof.** Maximizing revenue for a distribution with finite support is a linear programming problem (the unknowns are the \( q_i(x) \) and \( s(x) \) for all \( x \) in the support, the constraints are the IR and IC inequalities, and the objective function is the expected revenue). Using MAPLE yields the following.\(^{25}\)

The (unique\(^{26}\)) optimal mechanism for \( F_2 \times F_2 \) consists of 11 outcomes; see Figure 2 (the outcomes are ordered according to increasing payment to the seller \( s \)). For \( F_1 \times F_1 \) the same mechanism is optimal; however, the fifth

<table>
<thead>
<tr>
<th>Valuations ( x )</th>
<th>Outcome ( q(x) )</th>
<th>Outcome ( s(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((10, 10), (10, 13), (13, 10), (13, 13), (10, 46), (46, 10), (13, 46), (46, 13), (46, 46), (13, 47), (47, 13), (10, 47), (47, 10))</td>
<td>((0, 0))</td>
<td>(0)</td>
</tr>
<tr>
<td>((46, 47))</td>
<td>(\left(\frac{32}{1187}, \frac{384}{13057}\right))</td>
<td>(34240 \approx 2.6)</td>
</tr>
<tr>
<td>((47, 46))</td>
<td>(\left(\frac{384}{13057}, \frac{32}{1187}\right))</td>
<td>(34240 \approx 2.6)</td>
</tr>
<tr>
<td>((47, 47))</td>
<td>(\left(\frac{35}{1187}, \frac{35}{1187}\right))</td>
<td>(3258 \approx 2.7)</td>
</tr>
<tr>
<td>((13, 80))</td>
<td>(\left(\frac{32}{1187}, \frac{5647}{5935}\right))</td>
<td>(90672 \approx 76.4)</td>
</tr>
<tr>
<td>((80, 13))</td>
<td>(\left(\frac{5647}{5935}, \frac{32}{1187}\right))</td>
<td>(90672 \approx 76.4)</td>
</tr>
<tr>
<td>((46, 80))</td>
<td>(\left(\frac{35}{1187}, \frac{5647}{5935}\right))</td>
<td>(90810 \approx 76.5)</td>
</tr>
<tr>
<td>((80, 46))</td>
<td>(\left(\frac{5647}{5935}, \frac{35}{1187}\right))</td>
<td>(90810 \approx 76.5)</td>
</tr>
<tr>
<td>((10, 80), (10, 100), (13, 100))</td>
<td>((0, 1))</td>
<td>(80)</td>
</tr>
<tr>
<td>((80, 10), (100, 10), (100, 13))</td>
<td>((1, 0))</td>
<td>(80)</td>
</tr>
<tr>
<td>((46, 100), (100, 46), (47, 80), (80, 47), (47, 100), (100, 47), (80, 80), (80, 100), (100, 80), (100, 100))</td>
<td>((1, 1))</td>
<td>(126)</td>
</tr>
</tbody>
</table>

Figure 2: The unique optimal mechanism for \( F_2 \times F_2 \)

and sixth outcomes are not used (the value 13 has probability 0) and may

\(^{25}\)The fractions appearing in the solutions below are exact.

\(^{26}\)Uniqueness is proved using the dual linear programmng problem, as in Section 2.2.
be dropped. This yields:

\[ \text{Rev}(F_1 \times F_1) = \frac{408189937}{5875650} \quad \text{and} \quad \text{Rev}(F_2 \times F_2) = \frac{30614162731}{440673750}, \]

and so indeed \( \text{Rev}(F_1 \times F_1) > \text{Rev}(F_2 \times F_2) \) (these revenues are 69.47145... and 69.47126...). The nonmonotonicity of the payments can be seen at 27

\[ s(10, 80) > s(13, 80), s(46, 80) \quad \text{and} \quad s(80, 10) > s(80, 13), s(80, 46). \]

\section*{3 Lotteries and Revenue}

To maximize revenue in the one-good case (i.e., \( k = 1 \)), it suffices to consider deterministic mechanisms (specifically, “posted-price” mechanisms; see Myerson 1981). That is not so in the multi-good case. Examples where the optimal mechanism requires randomization (i.e., in some of the outcomes the probability of getting a good is strictly between 0 and 1) have been provided by Thanassoulis (2004) (in the slightly different context of “unit demand,” where the buyer is limited to one good), Pycia (2006), Manelli and Vincent (2006, 2007), Briest, Chawla, Kleinberg, and Weinberg (2010) (for unit demand), and Pavlov (2011, Example 3(ii)). However, most of these examples are relatively complicated and require non-trivial computations, and it is not clear how and why randomization helps only when there are multiple goods.

We will provide two examples that are simple and transparent enough that the need for randomization becomes clear. In the first, the values of the two goods are correlated; in the second, the values are independent and identically distributed.28

\footnotetext{27}{Recall however that \( s \) being a nonmonotonic function, while necessary, is not sufficient; cf. the second paragraph of this section.}

\footnotetext{28}{Manelli and Vincent (2007) provide an example (Example 1) of an “undominated mechanism” that uses lotteries. While they prove that an undominated mechanism is optimal for some distribution \( F \), it is also claimed there (Theorem 9) that any undominated mechanism is optimal for some distribution with \textit{independent goods} (i.e., a product distribution). However, there is an error in the proof of Theorem 9, as the set of product distributions (specifically, the set \( G \) in their proof) is not convex. See the corrigendum in Manelli and Vincent (2012).}
3.1 Lotteries for multiple goods

Consider the following example with two goods and three possible valuations\(^{29}\) (the values of the two goods are correlated).

**Example E3.** Let \(\mathcal{F}\) be the following two-dimensional probability distribution:

\[\mathcal{F} = \begin{cases} (1, 0), & \text{with probability } 1/3, \\ (0, 2), & \text{with probability } 1/3, \\ (3, 3), & \text{with probability } 1/3. \end{cases}\]

**Proposition 10** The mechanism \((q, s)\) defined by

<table>
<thead>
<tr>
<th>Valuation (x)</th>
<th>Outcome (q(x))</th>
<th>Outcome (s(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 0))</td>
<td>((1, 0))</td>
<td>(1/2)</td>
</tr>
<tr>
<td>((0, 2))</td>
<td>((0, 1))</td>
<td>(2)</td>
</tr>
<tr>
<td>((3, 3))</td>
<td>((1, 1))</td>
<td>(5)</td>
</tr>
</tbody>
</table>

with buyer payoff function

\[b(x_1, x_2) = \max \left\{ 0, \frac{1}{2} x_1 - \frac{1}{2}, x_2 - 2, x_1 + x_2 - 5 \right\}\]

is the unique revenue-maximizing IC and IR mechanism for \(\mathcal{F}\) of Example E3.

Thus, the buyer can get both goods for price 5, or get good 2 for price 2, or get good 1 with probability 1/2 for price 1/2; the optimal revenue is 5/2 = 2.5.

If the seller were restricted to deterministic mechanisms (where each \(q_i\) is

\(^{29}\)Pycia (2006) solves the seller’s problem when there are exactly two valuations and shows that randomization may be needed. For instance, when the valuations are \((2, 3)\) and \((6, 1)\) with equal probabilities, the *unique* optimal mechanism gives the \((2, 3)\)-buyer good 2 and a 1/2 chance of getting good 1, for the total price of 4; and gives the \((6, 1)\)-buyer both goods for the total price of 7. However, we have found that Example E3, with three possible valuations, provides more transparent insights (as there is a clearer separation between the IC and IR constraints).
either 0 or 1), then the optimal revenue would decrease to $7/3 = 2.33...$
(attained for instance by selling separately, at the optimal single-good prices
of 3 for good 1 and 2 for good 2; see below). A detailed explanation of the
role of randomization, and why it is needed only when there are multiple
goods, follows the proof below.

**Proof.** Let $\langle (\alpha_1, \beta_1); \sigma_1 \rangle$, $\langle (\alpha_2, \beta_2); \sigma_2 \rangle$, and $\langle (\alpha_3, \beta_3); \sigma_3 \rangle$ be the outcome
$\langle (q_1(x), q_2(x); s(x)) \rangle$ at $x = (1, 0), (0, 2),$ and $(3, 3),$ respectively (thus $\alpha_i, \beta_i \in [0, 1]$). The objective function is $S := \sigma_1 + \sigma_2 + \sigma_3$ (this is 3 times the
revenue). Consider the relaxed problem of maximizing $S$ subject only to the
individual-rationality constraints at $(1, 0)$ and $(0, 2),$ and to the two incentive-
compatibility constraints at $(3, 3)$, i.e.,

\[
\begin{align*}
\alpha_1 - \sigma_1 & \geq 0, \\
2\beta_2 - \sigma_2 & \geq 0, \\
3\alpha_3 + 3\beta_3 - \sigma_3 & \geq 3\alpha_1 + 3\beta_1 - \sigma_1, \\
3\alpha_3 + 3\beta_3 - \sigma_3 & \geq 3\alpha_2 + 3\beta_2 - \sigma_2.
\end{align*}
\]

These inequalities can be rewritten as:

\[
\begin{align*}
\sigma_3 + 3\alpha_1 + 3\beta_1 - 3\alpha_3 - 3\beta_3 & \leq \sigma_1 \leq \alpha_1, \\
\sigma_3 + 3\alpha_2 + 3\beta_2 - 3\alpha_3 - 3\beta_3 & \leq \sigma_2 \leq 2\beta_2.
\end{align*}
\]

Therefore, in order to maximize $S = \sigma_1 + \sigma_2 + \sigma_3$ we must take $\sigma_1 = \alpha_1$ and
$\sigma_2 = 2\beta_2,$ which gives:

\[
\begin{align*}
\sigma_3 & \leq 3\alpha_3 + 3\beta_3 - 2\alpha_1 - 3\beta_1, \\
\sigma_3 & \leq 3\alpha_3 + 3\beta_3 - 3\alpha_2 - \beta_2.
\end{align*}
\]

Thus we must take $\alpha_3 = \beta_3 = 1,$ $\beta_1 = \alpha_2 = 0,$ and then $\sigma_3 = \min\{6 - 2\alpha_1, 6 - \beta_2\},$ and so $S = \alpha_1 + 2\beta_2 + \min\{6 - 2\alpha_1, 6 - \beta_2\} = \min\{2\beta_2 - \alpha_1, \beta_2 + \alpha_1\} + 6.$ Since $S$ is increasing in $\beta_2$ we must take $\beta_2 = 1,$ and then
$S = \min\{2 - \alpha_1, 1 + \alpha_1\} + 6$ is maximized at $\alpha_1 = 1/2$. This is precisely the mechanism (12), which is easily seen to satisfy also all the other IR and IC constraints.

To understand the use of randomization, consider the outcome $\langle(1/2, 0); 1/2\rangle$ at $x = (1, 0)$ in (12): it is a lottery ticket that costs $1/2$ and gives a $1/2$ probability of getting good 1; alternatively, it is a $1/2 - 1/2$ lottery between getting good 1 for the price 1 (i.e., $\langle(1, 0); 1\rangle$) and getting nothing and paying nothing (i.e., $\langle(0, 0); 0\rangle$). It is thus the average of these two deterministic outcomes, and we now consider what happens when we replace the lottery by either one of them (see Table 1 below). It turns out that in both cases the revenue strictly decreases. In the first case, replacing $\langle(1/2, 0); 1/2\rangle$ by $\langle(1, 0); 1\rangle$ forces the price of the bundle to decrease to 4 (otherwise the $(3, 3)$-buyer would switch from paying 5 for the bundle to paying 1 for good 1); therefore the net change in the revenue is $1/3 \cdot (1 - 1/2) + 1/3 \cdot (4 - 5)$, which is negative. In the second case, replacing $\langle(1/2, 0); 1/2\rangle$ by $\langle(0, 0); 0\rangle$ results in the loss of the revenue from the $(1, 0)$-buyer, without, however, increasing the revenue from the $(3, 3)$-buyer: indeed, if we were to increase the bundle price, then $(3, 3)$ would switch to $\langle(0, 1); 2\rangle$, i.e., would get good 2 for price 2 (and, if we were to drop this outcome $\langle(0, 1); 2\rangle$ altogether in order to increase the bundle price to 6, the total revenue would again decrease).

<table>
<thead>
<tr>
<th>$x$</th>
<th>$q(x)$</th>
<th>$s(x)$</th>
<th>$q^{(1)}(x)$</th>
<th>$s^{(1)}(x)$</th>
<th>$q^{(2)}(x)$</th>
<th>$s^{(2)}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 0 )</td>
<td>$(1/2, 0)$</td>
<td>$1/2$</td>
<td>(1, 0)</td>
<td>1</td>
<td>(0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>(0, 2 )</td>
<td>(0, 1)</td>
<td>2</td>
<td>(0, 1)</td>
<td>2</td>
<td>(0, 1)</td>
<td>2</td>
</tr>
<tr>
<td>(3, 3 )</td>
<td>(1, 1)</td>
<td>5</td>
<td>(1, 1)</td>
<td>4</td>
<td>(1, 1)</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 1: Replacing a lottery outcome when there are two goods

30For deterministic mechanisms (i.e., $\alpha_i, \beta_i \in \{0, 1\}$), everything is the same up to this point, but now $S$ is maximized at both $\alpha_1 = 0$ and $\alpha_1 = 1$; the optimal revenue for deterministic mechanisms is thus $S/3 = 7/3$.

31Because of risk neutrality.

32The buyer’s payoff function in this mechanism is $b^{(1)}(x) = \max\{x_1 - 1, x_2 - 2, x_1 + x_3 - 4\}$.

33The buyer’s payoff function in this mechanism is $b^{(2)}(x) = \max\{0, x_2 - 2, x_1 + x_2 - 5\}$.
It is instructive to compare this with a similar example, but with a single good. Assume the values are $x = 1, 0, 3$, with equal probabilities of $1/3$ each (just like good 1 in Example E3). Take the mechanism with outcomes $(1/2; 1/2), (0; 0), (1; 2)$ (see Table 2 below); it is easy to see that it is IC and IR, and its revenue is $5/6$. The lottery outcome $(1/2; 1/2)$—getting the good with probability $1/2$ for price $1/2$—is the average of $(0; 0)$ and $(1; 1)$. Replacing the lottery $(1/2; 1/2)$ with $(1; 1)$ lowers the revenue to $2/3$: the 3-buyer switches to $(1; 1)$. Replacing the lottery $(1/2; 1/2)$ with $(0; 0)$ increases the revenue to $1$: the 3-buyer is now offered, and chooses, $(1; 3)$. The revenue of $5/6$ of the original mechanism, which used the lottery outcome, is precisely the average of the revenues from these two resulting mechanisms, $2/3$ and $1$ (this averaging property holds at each valuation $x$).

<table>
<thead>
<tr>
<th>$x$</th>
<th>$q$</th>
<th>$s$</th>
<th>$q^{(1)}$</th>
<th>$s^{(1)}$</th>
<th>$q^{(2)}$</th>
<th>$s^{(2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/2</td>
<td>1/2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 2: Replacing a lottery outcome when there is one good

This is a general phenomenon when there is only one good: the revenue from a mechanism that includes an outcome that is a probabilistic mixture of two outcomes (a “lottery outcome”) is the average of the revenues obtained by replacing the lottery with each one of these two outcomes and then adapting the remaining outcomes. Formally, this is the counterpart of expressing the corresponding buyer payoff function $b$ as an average of two such functions; in the example above, $b(x) = \max\{0, x/2 - 1/2, x - 2\}$ is the $1/2 - 1/2$ average of $b^{(1)}(x) = \max\{0, x - 1\}$ and $b^{(2)}(x) = \max\{0, x - 3\}$ (i.e., $b(x) = (b^{(1)}(x) + b^{(2)}(x))/2$ for all $x$). Thus, lotteries are indeed not needed when there is only one good.

---

This statement, which is easily proved in general—even when the two outcomes that are averaged are not necessarily deterministic—provides another proof of Myerson’s result that in the one-good case it suffices to consider deterministic mechanisms (use this “local decomposition” repeatedly).
Example E3 illustrates why this is not the case for multiple goods: replacing the lottery outcome with \((0, 0); 0\) yields the mechanism \((q^{(2)}, s^{(2)})\), whose revenue is lower than that of \((q, s)\) (whereas replacing \((1/2; 1/2)\) with \((0; 0)\) yields a higher revenue). In fact, the function \(b\) of (13) is an extreme point in the set of buyer payoff functions (in particular, it is not the average of the buyer functions in footnotes 32 and 33).

This is exactly where having more than one good matters. In the case of one good there is only one binding constraint per value \(x\), namely, the outcome chosen by the next lower value. Consequently, removing an outcome (such as a lottery outcome) that is chosen by \(x\) enables the seller to increase the revenue obtained from all higher-valuation buyers (i.e., with values \(y > x\)), as they can no longer switch to the outcome that has been removed and they strictly prefer their own outcome to any of the outcomes chosen by values below \(x\). In contrast, when there are multiple goods, such an increase in revenue may not be possible because there may be multiple binding constraints for each valuation \(x\) (in our example, buyer \((3, 3)\) is indifferent between reporting truthfully and reporting either \((1, 0)\) or \((0, 2)\)). These buyer types may switch to other outcomes that involve other goods, and so the total revenue may well decrease.

Next, how does a lottery outcome increase revenue? The seller would like to earn positive revenue from selling good 1 to the \((1, 0)\) buyer, but without jeopardizing the higher revenue obtained from selling the bundle of both goods to the \((3, 3)\) buyer (and, as we have seen, he cannot increase the price of the bundle because of the “good 2 for price 2” alternative, i.e., \((0, 1); 2\)). If the price of good 1 is above 1 then \((1, 0)\) will not buy it; if it is below 1, then \((3, 3)\) will switch from buying the bundle to buying good 1 (since his payoff will increase from 1 to 2 or more).\(^{35}\) Thus selling good 1 does not help. What does help is selling only a fractional part of good 1, which has the effect of making this option less attractive to the high-valuation buyer \((3, 3)\) (since his possible gain is smaller: it is only that fraction of the difference in values). Thus, the two conflicting desiderata—getting some revenue from a

\(^{35}\) As we saw above, lowering the price of the bundle to 4 (while keeping the price of good 1 at 1) will not help either, because the total revenue decreases.
low-valuation buyer, and not jeopardizing the higher revenue from a higher-
valuation buyer—are reconciled by offering to sell fractions of the goods, i.e.,
lotteries. In the present example, that optimal fraction turns out to be 1/2;
it comes from balancing the incentives between the two goods (specifically,
1/2 is the ratio of two value differences, 3 – 2 for good 2 and 3 – 1 for good
1; see the Proof of Proposition 10 above).[36]

Finally, we note that mechanism design is a sequential game, with the
seller moving first. In such games, the use of randomization may in general
be strictly advantageous to the first mover (take for instance the sequential
“matching pennies” game). Thus, the surprising fact here is not that ran-
donization can increase revenue (when there are multiple goods), but that
it cannot do so when there is only one good.[37, 38]

3.2 Lotteries for independent and identically distributed
goods

We now provide a simple example where lotteries are necessary to achieve
the maximal revenue for two goods that are independent and identically
distributed.

Example E4. Let $F$ be the following one-dimensional probability distribu-
tion:

$$F = \begin{cases}
1, & \text{with probability } 1/6, \\
2, & \text{with probability } 1/2, \\
4, & \text{with probability } 1/3,
\end{cases}$$

and take two independent $F$-distributed goods, i.e., $\mathcal{F} = F \times F$.

[36] Thus one can easily get other probabilities by changing the values. Moreover, the
elementary is highly robust: it has a large neighborhood of distributions for which any
optimal mechanism requires lotteries.

[37] We thank Bob Aumann for this comment.

[38] Pycia (2006) shows how in the multiple-goods case non-deterministic mechanisms are
generically needed to maximize revenue.
Proposition 11 The mechanism \((q, s)\) defined by

\[
\begin{array}{|c|c|c|}
\hline
\text{Valuations} & \text{Outcome} \\
\hline
x & q(x) & s(x) \\
\hline
(1, 1) & (0, 0) & 0 \\
(2, 1) & (\frac{1}{2}, 0) & 1 \\
(1, 2) & (0, \frac{1}{2}) & 1 \\
(1, 4), (4, 1), (2, 2), (2, 4), (4, 2), (4, 4) & (1, 1) & 4 \\
\hline
\end{array}
\] (14)

with buyer payoff function

\[
b(x_1, x_2) = \max\left\{0, \frac{1}{2}x_1 - 1, \frac{1}{2}x_2 - 1, x_1 + x_2 - 4\right\}
\] (15)

is the unique optimal mechanism for \(\mathcal{F} = F \times F\) of Example E4.

Proof. First, the revenue from the mechanism (14) is easily computed: it is 61/18.

Second, consider the following inequalities, which are various individual
rationality and incentive compatibility constraints\textsuperscript{39}:

\[
\begin{align*}
q_1^{11} + q_2^{11} - s^{11} & \geq 0 & 3 \\
n_1^{12} + 2n_2^{12} - s^{12} & \geq 0 & 8 \\
n_1^{21} + q_2^{21} + s^{21} & \geq 0 & 8 \\
n_1^{22} + 2n_2^{22} - s^{22} & \geq 0 & 17 \\
n_1^{12} + 2n_2^{12} - s^{12} & \geq q_1^{11} + 2q_2^{11} - s^{11} & 1 \\
n_1^{21} + q_2^{21} - s^{21} & \geq 2q_1^{11} + q_2^{11} - s^{11} & 1 \\
n_1^{22} + 2q_2^{22} - s^{22} & \geq 2q_1^{12} + q_2^{12} - s^{12} & 3 \\
n_1^{22} + 2q_2^{22} - s^{22} & \geq 2q_1^{21} + 4q_2^{21} - s^{21} & 3 \\
q_1^{14} + 4q_2^{14} - s^{14} & \geq q_1^{13} + 4q_2^{13} - s^{13} & 3 \\
4q_1^{41} + q_2^{41} - s^{41} & \geq 4q_1^{21} + s^{21} - s^{21} & 3 \\
n_1^{22} + 2q_2^{22} - s^{22} & \geq 2q_1^{14} + 2q_2^{14} - s^{14} & 1 \\
n_1^{22} + 2q_2^{22} - s^{22} & \geq 2q_1^{41} + q_2^{41} - s^{41} & 1 \\
n_1^{24} + 4q_2^{24} - s^{24} & \geq 2q_1^{22} + 4q_2^{22} - s^{22} & 8 \\
4q_1^{12} + 2q_2^{12} - s^{22} & \geq 4q_1^{22} + s^{22} & 8 \\
4q_1^{44} + 4q_2^{44} - s^{44} & \geq 4q_1^{24} + 4q_2^{24} - s^{24} & 2 \\
4q_1^{44} + 4q_2^{44} - s^{44} & \geq 4q_1^{12} + 4q_2^{12} - s^{12} & 2 \\
\end{align*}
\]

Multiplying each inequality by the weight on the right and adding up yields:

\[
s^{11} + 3s^{12} + 3s^{21} + 9s^{22} + 2s^{14} + 2s^{41} + 6s^{24} + 6s^{42} + 4s^{44} \\
\leq 2n_2^{22} + q_1^{14} + 10q_1^{41} + 8q_1^{24} + 24q_1^{22} + 16q_1^{14} \\
+ 2q_2^{22} + 10q_2^{14} + q_2^{41} + 24q_2^{24} + 8q_2^{42} + 16q_2^{44}.
\]

The left-hand side turns out to be precisely 36 times the expected revenue of the seller for the distribution $\mathcal{F} = F \times F$, i.e., $36\mathbb{E}_\mathcal{F}[s(x)]$, and the right-hand side is bounded from above by 122 (replace all $q_1$ and $q_2$ there by their upper bound of 1). Therefore $\mathbb{E}_\mathcal{F}[s(x)] \leq 122/36 = 61/18$. Recalling that 61/18 is precisely the revenue of the mechanism (14) shows that (14) is optimal.

Finally, to see that (14) is the only optimal mechanism: by the proof above, for the maximal revenue of 61/18 to be achieved, all the inequalities

\textsuperscript{39}These specific inequalities and their corresponding multipliers below were obtained by solving the dual of the linear programming problem of maximizing revenue.
must become equalities. First, all the \( q_1 \) and \( q_2 \) appearing on the right-hand side of (17) must equal 1:

\[
1 = q_1^{22} = q_1^{14} = q_1^{41} = q_1^{24} = q_4^{42} = q_4^{44} \\
= q_2^{22} = q_2^{14} = q_2^{41} = q_2^{24} = q_2^{42} = q_2^{44} .
\] (18)

Second, the inequalities in (16), which are now equalities, yield, after substituting (18),

\[
s_{44} = s_{24} = s_{42} = s_{22} = s_{14} = s_{41} = 4, \\
s_{12} = s_{21} = 1, \\
s_{11} = 0, \\
q_{11}^{11} = q_{21}^{11} = q_{12}^{11} = q_{22}^{11} = 0, \\
q_{21}^{21} = q_{22}^{21} = \frac{1}{2}.
\]

Together with (18) this yields precisely the mechanism (14).

It can be checked that the maximal revenue achievable by a deterministic mechanism is \( 10/3 \) (obtained by the mechanism with price 2 for each good).

4 Seller-Favorable Mechanisms

When maximizing revenue one may without loss of generality consider only mechanisms that are “seller-favorable,” a concept that we introduced in Section 2.3. These are incentive-compatible mechanisms such that there is no other incentive-compatible mechanism for which the payoffs to the buyer do not change, while the payments to the seller do not decrease, and strictly increase in at least one case. We will see that the restriction to seller-favorable mechanisms simplifies the analysis (in particular, it circumvents nondifferentiability issues). Moreover, as we saw in Section 2.3, it is needed to obtain monotonicity results.

The characterization of IC mechanisms \((q, s)\) as those whose assignment function, \( q \), is a subgradient of the buyer’s convex payoff function is well known (starting with Rochet 1985). It is an inconvenient and often technically annoying fact that the buyer’s convex payoff function, while differentiable almost everywhere, need not be differentiable everywhere. Proofs that
are otherwise simple and elegant often require detours through subgradient measurable selection arguments.\textsuperscript{40}

The purpose of this section is to show that such detours can be avoided when one restricts attention to seller-favorable mechanisms. The reason is that the buyer’s payoff function is not differentiable only when he is indifferent between a number of reports. But if the mechanism \((q, s)\) is seller-favorable, the buyer’s truthful report must maximize the seller’s payoff from among all of the buyer’s optimal reports. As we show, this implies that \(q(x) \cdot x = b'(x; x)\), which denotes the directional derivative of \(b\) at \(x\) in the direction \(x\) (see the formal definitions after the proof of Lemma 12 below), for every buyer valuation \(x\). Consequently, in a seller-favorable mechanism the buyer’s payoff function \(b\) completely determines the seller’s payoff function \(s\) at every \(x\), whether it is a point of differentiability of \(b\) or not, and \(s(x) = b'(x; x) - b(x)\) for all \(x\).

To be as general as possible, we will work here with an arbitrary domain \(D \subset \mathbb{R}^k\) of valuations; \(D\) could be \(\mathbb{R}_+^k\), or it may be finite or infinite, and in general need not be convex or even connected. A mechanism is thus \((q, s) : D \to [0, 1]^k \times \mathbb{R}\), and the buyer’s payoff function is \(b(x) = q(x) \cdot x - s(x)\). The range \(M := (q, s)(D) = \{(q(x), s(x)) : x \in D\}\) of the mechanism, also called the menu of the mechanism, consists of all those combinations of outcomes \(g \in [0, 1]^k\) and payments \(t \in \mathbb{R}\) that are used \((M\) is a subset of \([0, 1]^k \times \mathbb{R}\)). The mechanism is incentive-compatible (IC) if \(b(x) = \max_{y \in D} (q(y) \cdot x - s(y)) = \max_{(g,t) \in M} (g \cdot x - t)\) for every \(x \in D\). It is seller-favorable if there is no other incentive-compatible mechanism \((\tilde{q}, \tilde{s})\) on \(D\) having the same buyer payoff function, i.e., \(\tilde{q}(x) \cdot x - \tilde{s}(x) = b(x) = q(x) \cdot x - s(x)\) for all \(x \in D\), and a larger payment function, i.e., \(\tilde{s}(x) \geq s(x)\) for every \(x \in D\), with strict inequality for some \(x \in D\). This implies, in particular, that when the buyer is indifferent, ties must be broken in favor of the seller, i.e., \(q(y) \cdot x - s(y) = q(x) \cdot x - s(x)\) implies \(s(y) \leq s(x)\).

The first lemma shows that one may extend any IC mechanism to a larger domain, even all \(\mathbb{R}^k\), and without increasing the menu \(M\) beyond its closure, which we denote by \(\text{cl} M\).

\textsuperscript{40}E.g., Lemma A.4 in Manelli and Vincent (2007); see footnote 51 below.
Lemma 12 Let \((q, s)\) be an IC mechanism on a domain \(D \subset \mathbb{R}^k\), with menu \(M = (q, s)(D)\). Then \((q, s)\) can be extended to an IC mechanism \((\bar{q}, \bar{s})\) on the whole space \(\mathbb{R}^k\), with menu \(\bar{M} = (\bar{q}, \bar{s})(\mathbb{R}^k)\) that satisfies \(M \subset \bar{M} \subset \text{cl} M\).

Proof. Defining \(b(x) := \sup_{(g, t) \in M} (g \cdot x - t)\) for every \(x \in \mathbb{R}^k\) extends the buyer’s payoff function from \(D\) to \(\mathbb{R}^k\). The function \(s\) is bounded from below (since \(b(x) = q(x) \cdot x - s(x)\) is finite for \(x \in D\)), and so there is \(\tau\) such that \(t \geq \tau\) for all \((g, t) \in M\). Fix some element \((g_0, t_0) \in M\); then, for every \(x \in \mathbb{R}^k\), only those \((g, t)\) in \(M\) with \(t \leq ||x||_1 + t_0\) matter for \(b(x)\) (since \(g \cdot x - t \geq g_0 \cdot x - t_0\) implies \(t \leq (g - g_0) \cdot x + t_0 \leq ||x||_1 + t_0\)). Therefore for every \(x \in \mathbb{R}^k\) the supremum in the definition of \(b(x)\) is attained, say at \((\bar{q}(x), \bar{s}(x))\) \(\in \text{cl} M\); for \(x \in D\) we take \((\bar{q}(x), \bar{s}(x)) = (q(x), s(x))\). Thus \(b(x) = \bar{q}(x) \cdot x - \bar{s}(x) = \max_{(g, t) \in M} (g \cdot x - t) = \max_{y \in \mathbb{R}^k} (\bar{q}(y) \cdot x - \bar{s}(y))\) for every \(x \in \mathbb{R}^k\), which says that \((\bar{q}, \bar{s})\) is IC on \(\mathbb{R}^k\). ■

We shall thus consider without loss of generality mechanisms on \(\mathbb{R}^k\). The buyer’s payoff function \(b\), as the pointwise supremum of affine functions, is a convex function on \(\mathbb{R}^k\), finite everywhere (since, as we have seen, for every \(x \in \mathbb{R}^k\) the supremum is attained).

We recall a few useful concepts for convex functions.\(^{41}\) Let \(f : \mathbb{R}^k \to \mathbb{R}\) be a real convex function defined on \(\mathbb{R}^k\) (and so \(\text{dom} f = \mathbb{R}^k\)). The directional derivative of \(f\) at \(x \in \mathbb{R}^k\) in the direction \(y \in \mathbb{R}^k\) is \(f'(x; y) := \lim_{\delta \to 0^+} (f(x + \delta y) - f(x))/\delta\). Since \(f\) is convex, \(f'(x; y)\) always exists. A vector \(g \in \mathbb{R}^k\) is a subgradient of \(f\) at \(x \in \mathbb{R}^k\) if \(f(y) - f(x) \geq g \cdot (y - x)\) for all \(y \in \mathbb{R}^k\). The set \(\partial f(x)\) of subgradients of \(f\) at \(x\) is a nonempty compact set, and \(f'(x; y) = \max\{g \cdot y : g \in \partial f(x)\}\) for every \(x, y \in \mathbb{R}^k\). Finally, if \(0 \leq f(x + z) - f(x) \leq \sum_{i=1}^{k} z_i\) holds for every \(x, z \in \mathbb{R}^k\) with \(z \geq 0\), then the function \(f\) is nondecreasing and nonexpansive.\(^{42}\)

Let \(B^k\) be the collection of all real functions on \(\mathbb{R}^k\) that are nondecreasing, nonexpansive, and convex.

\(^{41}\)See Rockafellar (1970) for convex functions, their derivatives, and (sub)gradients.

\(^{42}\)For convex \(f\), this is equivalent to \(0 \leq \partial f(x)/\partial x_i \leq 1\) for all \(i\) and all \(x\) where the derivative exists (i.e., for a.e. \(x\)).
Lemma 13 Let \((q, s)\) be an IC mechanism on \(\mathbb{R}^k\). Then the buyer’s payoff function \(b\) belongs to \(\mathcal{B}^k\), and, for every \(x \in \mathbb{R}^k\), the vector \(q(x)\) is a subgradient of \(b\) at \(x\) and \(s(x) \leq b'(x; x) - b(x)\).

Proof. Let \(M = (q, s)(\mathbb{R}^k)\) be the menu of \((q, s)\); then \(b(x) = q(x) \cdot x - s(x) = \max_{(g, t) \in M} (g \cdot x - t)\), as a supremum of affine functions, is a convex function. For every \(x, y \in \mathbb{R}^k\) we get

\[
b(y) - b(x) \geq (q(x) \cdot y - s(x)) - (q(x) \cdot x - s(x)) = q(x) \cdot (y - x),
\]

which says that \(q(x)\) is a subgradient of \(b\) at \(x\). Thus, \(q(x) \cdot x \leq \sup\{g \cdot x : g \in \partial b(x)\} = b'(x; x)\) and so \(s(x) = q(x) \cdot x - b(x) \leq b'(x; x) - b(x)\) for every \(x \in \mathbb{R}^k\). Finally, taking \(y = x + z\) with \(z \geq 0\) in (19) implies that \(0 \leq q(x) \cdot z \leq b(x + z) - b(x) \leq q(x + z) \cdot z \leq \sum_{i=1}^{k} z_i\), and so \(b\) is nondecreasing and nonexpansive.

Lemma 14 Let \(b\) be a function in \(\mathcal{B}^k\). Then there exists an IC mechanism \((q, s)\) on \(\mathbb{R}^k\) such that the buyer’s payoff function is \(b\).

Proof. Being nondecreasing, nonexpansive, and convex on \(\mathbb{R}^k\), the function \(b\) satisfies \(0 \leq b(x) - b(x - z) \leq g \cdot z \leq b(x + z) - b(x) \leq \sum_{i=1}^{k} z_i\) for every \(x \in \mathbb{R}^k\), every \(g \in \partial b(x)\), and every \(z \in \mathbb{R}^k_+\). In particular, \(\partial b(x) \subset [0, 1]^k\), and so we choose for each \(x\) some \(q(x) \in \partial b(x)\) and put \(s(x) := q(x) \cdot x - b(x)\). Then \(q(y) \cdot y - s(y) = b(y) \geq b(x) + q(x) \cdot (y - x) = q(x) \cdot y - s(x)\) (the inequality since \(q(x) \in \partial b(x)\)), and so \((q, s)\) is IC.

Proposition 15 Let \((q, s)\) be an IC mechanism on a domain \(D \subset \mathbb{R}^k\), with buyer payoff function \(b\) and menu \(M = (q, s)(D)\). Then there exists a seller-favorable IC mechanism \((\tilde{q}, \tilde{s})\) on \(\mathbb{R}^k\) such that:

(i) \(\tilde{q}(x) \cdot x - \tilde{s}(x) = b(x) = q(x) \cdot x - s(x)\) for every \(x \in D\).

\[\text{as the proof below shows, (i) and (ii) hold in fact for all } x \in \mathbb{R}^k\text{, with } b \text{ the buyer payoff function of the extension of } (q, s) \text{ from } D \text{ to } \mathbb{R}^k \text{ obtained by Lemma 12}. \text{ The equality in (ii) implies in particular that the payment function } \tilde{s} \text{ of a seller-favorable mechanism is a Borel-measurable function on } \mathbb{R}^k.\]
(ii) \( \tilde{s}(x) = b'(x; x) - b(x) \geq s(x) \) for every \( x \in D \).

(iii) \( \tilde{M} := (\tilde{q}, \tilde{s})(\mathbb{R}^k) \subseteq \text{cl}((q, s)(D)) = \text{cl} M \).

**Remark.** Since \( b'(x; x) = \max_{g \in \partial b(x)} g \cdot x \), choosing in the proof of Lemma 14 a \( \tilde{q}(x) \) in \( \partial b(x) \) where this maximum is attained yields \((\tilde{q}, \tilde{s})\) that satisfies (i) and (ii), and so it is seller-favorable (by Lemma 13). However, the additional conclusion (iii) that the menu does not change (up to closure) is needed in order to guarantee that certain properties of the mechanism, such as submodularity, are preserved by seller-favorability (as in Corollary 19 in Section 4.1 below); (iii) requires a somewhat more elaborate proof.

**Proof.** Applying Lemma 12 allows us to assume without loss of generality that the domain of \((q, s)\) is the whole space, i.e., \( D = \mathbb{R}^k \) (the result for the extended mechanism clearly implies the result for the original one; note that if \( \tilde{M} \) is the menu of the extended mechanism, then \( \tilde{M} \subseteq \text{cl} \tilde{M} \) implies \( \tilde{M} \subseteq \text{cl} M \) because \( \tilde{M} \subseteq \text{cl} M \).

For every \( x \in \mathbb{R}^k \), define \((\tilde{q}(x), \tilde{s}(x))\) to be any limit point of the bounded sequence of points \((q(x_n), s(x_n)) \in [0, 1]^k \times \mathbb{R} \), where \( x_n := (1 + 1/n)x \) for each positive integer \( n \) (the sequence \( s(x_n) \) is bounded because \( s(x_n) = q(x_n) \cdot x_n - b(x_n) \) and \( b \) is continuous). Thus \((\tilde{q}, \tilde{s})\) satisfies (i) (again, because \( b \) is continuous) and (iii), and it is IC because \((q, s)\) is IC. Now for any \( x \) and \( n \), Lemma 13 implies that \( \tilde{q}(x) \in \partial b(x) \) and \( q(x_n) \in \partial b(x_n) \). In particular,\(^{44}\)

for every \( g \in \partial b(x) \) we have \( 0 \leq (q(x_n) - g) \cdot (x_n - x) = (q(x_n) - g) \cdot x/n \). Multiplying by \( n \) and taking the limit gives \( \tilde{q}(x) \cdot x \geq g \cdot x \), and so \( \tilde{q}(x) \cdot x = \max_{g \in \partial b(x)} g \cdot x = b'(x; x) \). The equality in (ii) follows because \( \tilde{s}(x) = b'(x) \cdot x - b(x) = b'(x; x) - b(x) \) by (i), and the inequality in (ii) follows from Lemma 13, which also implies that \((\tilde{q}, \tilde{s})\) is seller-favorable. ■

It is useful to gather the above results into one theorem.

**Theorem 16** Let \((q, s) : D \rightarrow [0, 1]^k \times \mathbb{R} \) be a mechanism defined on a domain \( D \subseteq \mathbb{R}^k \), with menu \( M := (q, s)(D) = \{(q(x), s(x)) : x \in D\} \) and

\[\text{if } f \text{ is a convex function then } (g - g') \cdot (x - y) \geq 0 \text{ for all } x, y \text{ and all } g, g' \in \partial f(x) \text{ and } g' \in \partial f(y) \text{ (add the inequalities } f(y) - f(x) \geq g \cdot (y - x) \text{ and } f(x) - f(y) \geq g' \cdot (x - y)).\]
buyer payoff function\(^{45}\) \(b : \mathbb{R}^k \to \mathbb{R}\) given by \(b(x) := \sup_{y \in D}(q(y) \cdot x - s(y)) = \sup_{(g,t) \in M}(g \cdot x - t)\) for every \(x \in \mathbb{R}^k\). Then:

(i) \((q, s)\) is an IC mechanism if and only if it is the restriction to \(D\) of an IC mechanism \((\tilde{q}, \tilde{s})\) on \(\mathbb{R}^k\) with the same buyer payoff function \(b\), and menu \(\overline{M} := (\tilde{q}, \tilde{s})(\mathbb{R}^k)\) that satisfies \(M \subset \overline{M} \subset \text{cl} M\).

(ii) \((q, s)\) is an IC mechanism if and only if \(b \in B^k\) and, for every \(x \in D\), we have \(q(x) \in \partial b(x)\) and \(s(x) = q(x) \cdot x - b(x) \leq b'(x; x) - b(x)\).

(iii) \((q, s)\) is in addition seller-favorable if and only if for every \(x \in D\) we have \(s(x) = b'(x; x) - b(x)\).

(iv) If \((q, s)\) is IC then there is a seller-favorable IC mechanism \((\tilde{q}, \tilde{s})\) on \(\mathbb{R}^k\) with the same buyer payoff function \(b\), with menu \(\tilde{M} := (\tilde{q}, \tilde{s})(\mathbb{R}^k)\) satisfying \(\tilde{M} \subset \text{cl} M\), and such that \(\tilde{s}(x) = b'(x; x) - b(x) \geq s(x)\) for all \(x \in D\).

Consider now the problem of maximizing the seller’s expected revenue subject to individual rationality (IR) for the buyer (i.e., \(b \geq 0\)). Since, by Theorem 16 (iv), it is without loss of generality to restrict attention to seller-favorable mechanisms, we get the following.

**Corollary 17** The seller’s maximal expected revenue is

\[
\text{REV}(\mathcal{F}) = \sup_{b \in B^k, b \geq 0} E_F [b'(x; x) - b(x)].
\]  

**(20)**

Remarks. (a) **Full implementation of approximate seller-favorable payoffs.** Given an IC mechanism \((q, s)\), the buyer may have more than one optimal choice at some valuation \(x\), in which case the seller’s revenue may be above or below \(s(x)\). However, we can modify the mechanism so that all optimal choices of the buyer yield (almost) the seller-favorable payment. The idea is simple: the seller gives a small fixed proportional discount on all payments.

---

\(^{45}\)The buyer’s payoff function \(b\) is always taken to be defined on the whole space \(\mathbb{R}^k\).
Consequently, whenever the buyer was indifferent, he will now strictly prefer the choice with the higher payment (where he gets the higher discount).

Formally, consider an arbitrary IC mechanism $(q, s)$ on a domain $D$, let $M := (q, s)(D)$ be its menu, and take a small $\varepsilon > 0$. Let $(q_\varepsilon, s_\varepsilon)$ be an IC mechanism that is obtained from the menu of $(q, (1 - \varepsilon)s)$, i.e., the buyer’s payoff function is $b_\varepsilon(x) := \sup_{y \in D} (q(y) \cdot x - (1 - \varepsilon)s(y))$ and $(q_\varepsilon(x), s_\varepsilon(x)) \in \text{cl} \{(q(y), (1 - \varepsilon)s(y)) : y \in D\}$ for every $x \in D$. We claim that such a mechanism $(q_\varepsilon, s_\varepsilon)$ guarantees to the seller, for any optimal choice of the buyer at $x$, a payment of at least $(1 - \varepsilon)(b'(x; x) - b(x))$, i.e., a $(1 - \varepsilon)$-proportion of the seller-maximal payoff, for every $x \in D$. To see this, assume for simplicity that the menu $M$ is already a closed set, then given $x \in D$ there is $z \in D$ such that $(q(z), s(z))$ is the seller-maximal payoff at $x$, i.e., $b(x) = q(z) \cdot x - s(z)$ and $s(z) = b'(x; x) - b(x)$ (see Theorem 16 (iv)). Then, in $(q_\varepsilon, s_\varepsilon)$, for every optimal choice $(q(y), (1 - \varepsilon)s(y))$ at $x$ we have

$$q(y) \cdot x - (1 - \varepsilon)s(y) \geq q(z) \cdot x - (1 - \varepsilon)s(z) = b(x) + \varepsilon s(z) \geq q(y) \cdot x - s(y) + \varepsilon s(z)$$

(the last inequality by IC of $(q, s)$). Hence $s(y) \geq s(z)$ (subtract and divide by $\varepsilon$), and so the seller’s payoff, $s_\varepsilon(x) = (1 - \varepsilon)s(y)$, is indeed at least $(1 - \varepsilon)s(z) = (1 - \varepsilon)(b'(x; x) - b(x))$, i.e., $(1 - \varepsilon)$ of the seller-maximal payoff at $x$.

(b) Unique equilibrium. Given any IC mechanism $(q, s)$, there are various ways to eliminate, at arbitrarily small cost, the problem of the buyer having only weak incentives to report truthfully. For example, one can introduce an arbitrarily small positive probability that, after the buyer reports his valuation, the given (IC) mechanism is replaced by a random reserve price on each good (independently).48

---

46Thus the tie-breaking rule in favor of the seller is obtained as the limit of any optimal behavior of the buyer in the perturbed mechanisms. Moreover, this result holds starting with any IC mechanism $(q, s)$.

47Otherwise one takes appropriate sequences $z_n \in D$ and $y_n \in D$ instead of $z$ and $y$ below.

48Thus, in the case of a single buyer, the possibility of multiple optimal reports for the
exists if \( D \) in \([0, 1]^k \), i.e., \( g_\lambda := g + \lambda h \notin [0, 1]^k \) for every \( \lambda > 0 \). The reason is that if \( g_\lambda \in [0, 1]^k \) then the mechanism \((q', s')\) on \( D \) that is identical with \((q, s)\) except at \( x \), where we put \( q'(x) := g_\lambda \) and \( s'(x) := b(x) - g_\lambda \cdot x = s(x) - \lambda h \cdot x \), which is larger than \( s(x) \) (because \( h \cdot x > 0 \)), is IC (since for every \( y \in D \) we have \( q'(x) \cdot y - s'(x) \leq q(x) \cdot y - s(x) \) because \( h \cdot y \leq h \cdot x \)) and has the same buyer payoff function \( b \), which contradicts the seller favorability of \((q, s)\). If \( h \cdot x = 0 \) then \( s'(x) = s(x) \) in the construction above, and so we can assume without loss of generality that \( g \) is maximal in the direction \( h \).

In particular, we have:

- without loss of generality \( q_i(x) = 0 \) when \( x_i = 0 \) (take \( h = -e^{(i)} \), where \( e^{(i)} \in \mathbb{R}^k \) is the \( i \)-th unit vector);
- \( q_i(x) = 1 \) when \( x_i = \max\{y_i : y \in D\} > 0 \) (for instance, if \( D = [0, 1]^k \); then \( q_i(x) = 1 \) when \( x_i = 1 \); take \( h = e^{(i)} \));
- \( \max_i q_i(x) = 1 \) when \( \sum_i x_i = \max\{\sum_i y_i : y \in D\} > 0 \) (for instance, when \( D \) is the unit simplex in \( \mathbb{R}_{++}^k \); take \( h = (1, 1, \ldots, 1) \)).

(d) Nonnegative revenue. If \( b(0) = 0 \) (which, when maximizing revenue, can always be assumed when \( D \subset \mathbb{R}_+^k \)), then \( s(x) \geq 0 \) (i.e., there are no positive transfers from seller to buyer), and so \( b'(x; x) \geq b(x) \).

(e) \( s(x) = b'(x; x) - b(x) \) is the right-derivative of the function \( t \to b(tx) - tb(x) \) at \( t = 1 \) (because \( b'(x; x) = \lim_{\delta \to 0^+} (b((1 + \delta)x) - b(x))/\delta \) is the right-derivative of the function \( t \to b(tx) \) at \( t = 1 \); these functions relate to the local returns to scale of \( b \)).
4.1 Pricing functions and submodularity

A function \( p : [0, 1]^k \to \mathbb{R} \cup \{ \infty \} \) is a pricing function for the IC mechanism \((q, s)\) on the domain \( D \) if for all \( x \in D \) we have \( p(q(x)) = s(x) \) and\(^{52}\) \( b(x) = q(x) \cdot x - p(q(x)) = \max_{g \in [0, 1]^k} (g \cdot x - p(g)) \); thus, the choice function \((q, s)\) obtains from the menu consisting of all lotteries \( g \in [0, 1]^k \), each one priced at \( p(g) \). A pricing function always exists: take \( p(g) = s(x) \) for \( g = q(x) \in Q := q(D) \) and \( p(g) = \infty \) otherwise.

Lemma 18 Let \((q, s)\) be an IC mechanism on a domain \( D \subset \mathbb{R}^k \). Then the function \( p_0 : [0, 1]^k \to \mathbb{R} \cup \{ \infty \} \) given by \( p_0(g) := \sup_{x \in D} (g \cdot x - b(x)) \) for all \( g \in [0, 1]^k \) is the minimal pricing function for \((q, s)\); i.e., \( p_0 \) is a pricing function for \((q, s)\), and any pricing function \( p \) for \((q, s)\) satisfies \( p(g) \geq p_0(g) \) for all \( g \in [0, 1]^k \).

Proof. \( p_0 \) is a pricing function for \((q, s)\) since for \( g = q(x) \) with \( x \in D \) we have \( g \cdot x - p(g) = b(x) \) and \( q \cdot y - p(g) \leq b(y) \) for all \( y \in D \), and so the sup in the definition of \( p_0(g) \) is attained at \( x \) and equals \( s(x) \). If \( p \) is any pricing function then for each \( g \in [0, 1]^k \) we have \( g \cdot x - p(g) \leq b(x) \) for every \( x \in D \), and so \( \sup_{x \in D} (g \cdot x - b(x)) \leq p(g) \), i.e., \( p_0(g) \leq p(g) \). \( \blacksquare \)

We refer to this minimal \( p_0 \) as the canonical pricing function of \((q, s)\) (it is the Fenchel conjugate \( b^* \) of the buyer payoff function\(^{53}\) \( b : \mathbb{R}^k \to \mathbb{R} \)), in particular in view of the following.

\(^{52}\)Recall that if \( q(x) = q(y) \) then \( s(x) = s(y) \) (by IC). Thus the menu \( M = (q, s)(D) \) of \((q, s)\) is precisely the graph of the restriction of \( p \) to \( Q := q(D) \) (and the value \( p(g) \) for \( g \notin Q \) can be taken to be arbitrary in the range \( p_0(g) \leq p(g) \leq \infty \), where \( p_0 \) is given in Lemma 18).

\(^{53}\)It can be shown that \( p_0 \) is also the only closed convex pricing function \( p \) (since \( p^* = b = p_0^* \)). Convexity is a natural requirement on pricing functions, as a risk-neutral buyer can always randomize between choices and pay the expected price.
The mechanism \((q, s)\) is \textit{submodular} on \(D\) if it has a pricing function \(p\) that satisfies
\[
p(g) + p(g') \geq p(g \lor g') + p(g \land g')
\](21)
for all \(g, g' \in Q = q(D)\). In this case the minimal pricing function \(p_0\) also satisfies (21) (because, for \(p_0\), the left-hand side is the same, and the right-hand side can only be smaller by Lemma 18). Therefore \((q, s)\) is submodular if and only if its minimal pricing function \(p_0\) satisfies (21), i.e.,
\[
p_0(g) + p_0(g') \geq p_0(g \lor g') + p_0(g \land g')
\](22)
for all \(g, g' \in Q\).

We now show that submodularity is preserved for seller-favorable mechanisms with the same \(b\).

\textbf{Corollary 19} Let \((q, s)\) be a submodular IC mechanism on a domain \(D \subset \mathbb{R}^k\). Then there exists a seller-favorable submodular IC mechanism \((\tilde{q}, \tilde{s})\) on \(D\) with the same buyer payoff function (and so \(\tilde{s}(x) \geq s(x)\) for all \(x \in D\)).

\textbf{Proof.} Let \((\bar{q}, \bar{s})\) be the extension of \((q, s)\) to \(\mathbb{R}^k\) given by Theorem 16 (i), and apply Theorem 16 (iv) to \((\bar{q}, \bar{s})\) to obtain a seller-favorable IC mechanism \((\tilde{q}, \tilde{s})\). Since the buyer’s payoff function on \(D\) is the same for \((q, s)\) and \((\tilde{q}, \tilde{s})\), the corresponding function \(p_0\) is also the same. However, the set \(Q\) to which \(g\) and \(g'\) belong is now replaced by the set \(\tilde{Q} := \tilde{q}(D)\). We will now show that if \(p_0\) satisfies (22) on \(Q\) then it satisfies (22) also on \(\tilde{Q}\).

Let \(\tilde{g} = \tilde{q}(x) \in \tilde{Q}\). Theorem 16 (iv) implies that there is a sequence \((g_n, t_n) \in (q, s)(D)\) converging to \((\tilde{g}, \tilde{s}(x))\), and so, by the definition of \(p_0\), we have \(p_0(g_n) = t_n\) and \(p_0(\tilde{g}) = \tilde{g} : x - b(x) = \tilde{s}(x) = \lim_n t_n = \lim_n p_0(g_n)\) (in fact we have equality here, namely \(p_0(\tilde{g}) = \lim p_0(g_n)\), since \(p_0\) is a lower-semicontinuous function\footnote{\(p_0\), as the sup of a collection of affine—hence continuous—functions, is lower-semicontinuous: if \(g_n \rightarrow g\) then \(\lim \inf_n p_0(g_n) \geq p_0(g)\).}). Therefore, given \(\tilde{g}, \tilde{g}' \in \tilde{Q}\), take appropriate sequences \(g_n, g'_n \in Q\) such that \(g_n \rightarrow \tilde{g}, g'_n \rightarrow \tilde{g}'\), and then we have
\[
p_0(g_n) + p_0(g'_n) \geq p_0(g_n \lor g'_n) + p_0(g_n \land g'_n)
\]
for every $n$ by (22) on $Q$. The limit of the left-hand side is (at most) $p_0(\tilde{g}) + p_0(\tilde{g}')$, and the lim inf of the right-hand side is at least $p_0(\tilde{g} \lor \tilde{g}') + p_0(\tilde{g} \land \tilde{g}')$ by the lower-semicontinuity of $p_0$, which proves that $p_0$ satisfies (22) on $\tilde{Q}$.

References


