Strategic approximations of discontinuous games

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Abstract An infinite game is approximated by restricting the players to finite subsets of their pure strategy spaces. A strategic approximation of an infinite game is a countable subset of pure strategies with the property that limits of all equilibria of all sequences of approximating games whose finite strategy sets eventually include each member of the countable set must be equilibria of the infinite game. We provide conditions under which infinite games admit strategic approximations.

Keywords Discontinuous games · Finite approximation

JEL Classification C7

1 Introduction

The analytic convenience of infinite strategy spaces has often proven to be of value in the analysis of games. But when the presence of infinitely many strategies is crucial—such as when discontinuities play a central role and cannot be eliminated or smoothed—doubts may arise over the robustness or even the relevance of the results. One way to attenuate such doubts is to provide a sequence of finite approximating games whose equilibria converge to equilibria of the infinite game, where each approximating game restricts the players to finite subsets of their strategy spaces. This

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approach becomes even more convincing when there is some robustness in the choice of the approximation.

As observed by Simon (1987), a game with infinite strategy spaces might include strategies that are of particular strategic significance to the players. When this is the case, one cannot hope to well-approximate the infinite game without eventually including such strategies in the approximation. Thus, on the one hand, good approximations cannot always be arbitrary—they sometimes must include particular strategies. On the other hand, once such strategies are identified for eventual inclusion, the approximating games—i.e., the sequence of finite strategy sets—ought otherwise be arbitrary so as to establish the irrelevance of the remaining details of the approximating sequences.

A strategic approximation is a countable subset of pure strategies with the property that limits of all equilibria of all sequences of approximating games whose finite strategy sets eventually include each member of the countable set must be equilibria of the infinite game. Our objective is to provide conditions under which strategic approximations exist.²

2 Preliminaries

We maintain the following assumptions throughout the paper. There are $N$ players. Player $i$ has pure strategy set $X_i$, a nonempty compact metric space. We let $X = \times_i X_i$ and endow all product sets with the product topology. Player $i$'s von Neumann–Morgenstern utility function, $u_i : X \to \mathbb{R}$, is bounded and measurable. This defines a game $G = (X_i, u_i)_{i=1}^N$.

Let $M_i$ denote the space of probability measures on the Borel subsets of $X_i$ and let $M = \times_i M_i$. Extend each $u_i$ to $M$ by $u_i(m_1, \ldots, m_N) = \int_X u_i(x) dm_1 \cdots dm_N$. The space $M$ is a compact metric space when endowed with the Prohorov metric (see Billingsley 1968).

The mixed extension of $G$ is $\tilde{G} = (M_i, u_i)_{i=1}^N$. Let $\Delta(M_i)$ denote the space of probability measures on the Borel subsets of $M_i$ and let $\Delta(M) = \times_i \Delta(M_i)$. Extend each $u_i$ to $\Delta(M)$ by $u_i(\mu_1, \ldots, \mu_N) = \int_M u_i(m) \mu_1 \cdots \mu_N$. Like $M$, the space $\Delta(M)$ is also a compact metric space when endowed with the Prohorov metric. Given $\mu \in \Delta(M)$, define its distribution on $X$, $\bar{m} \in M$, by $\bar{m}(B) = \int_B \mu(B) \mu(m)$ for every Borel subset $B$ of $X$.³ We will use the fact that every mixed strategy $\mu \in \Delta(M)$ in the mixed extension of $G$ is payoff equivalent to its distribution $m$ on $X$, i.e., $u_i(\mu) = u_i(\bar{m})$ for every player $i$.⁴

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¹ Consider, for example, the strategy to produce zero for a firm with fixed costs that must be paid only when production is positive.

² A rather distinct approach to approximating infinite games can be pursued by exploiting the techniques introduced in Simon and Zame (1990) and redefining payoffs at points of discontinuity.

³ For fixed $B$, the real-valued function of $m$ defined by $m(B)$ is upper semicontinuous in $m$ on $M$ and hence measurable.

⁴ Indeed, by the definition of $\bar{m}$, $\int_X f(x) d\bar{m}(x) = \int_M \int_X f(x) dm(x) d\mu(m)$ for any Borel set $B$, where $I_B$ is its characteristic function. It follows that $\int_X f(x) d\bar{m}(x) = \int_M \int_X f(x) dm(x) d\mu(m)$ for every bounded measurable function $f$ and in particular for $f = u_i$. 

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By equilibrium we shall always mean a mixed strategy Nash equilibrium of the game under consideration. Thus, an equilibrium of $G$ lies in $M$, while an equilibrium of $G'$ lies in $\Delta(M)$.

$G' = (X'_i, u_i)_{i=1}^N$ is an approximation of $G = (X_i, u_i)_{i=1}^N$ if each $X'_i$ is a finite subset of $X_i$ and $v_i$ is the restriction of $u_i$ to $\times_i X'_i$. From now on, we will write $u_i$ instead of $v_i$, so that $G' = (X'_i, u_i)_{i=1}^N$. Thus, an approximating game simply restricts the players to finite subsets of their pure strategy sets. The following concept will figure prominently here.

**Definition 1** (Reny 2009). The game $G$ has the finite deviation property if whenever $m \in M$ is not an equilibrium of $G$, there is a neighborhood $U$ of $m$ and a finite subset $D$ of $M$ such that for every $m' \in U$ there is a player $i$ and $m' \in D$ such that $u_i(m_i, m'_{-i}) > u_i(m')$. The set $D$ is called a finite deviation set for $U$. If in addition the members of $D$ can always be chosen to have finite supports we say that $G$ has the finite-support finite deviation property.

Reny (2009) observes that if $G$ has the finite deviation property then $G$ possesses a mixed strategy equilibrium and demonstrates that $G$ has the finite deviation property if its mixed extension is better-reply secure. Because of the connection to better-reply security, we remind the reader of the definition.

A pair $(m, u) \in M \times \mathbb{R}^N$ is in $\Gamma$, the closure of the graph of the vector payoff function of $G$, if $u = \lim_n (u_1(m^n), \ldots, u_N(m^n))$ for some sequence $m^n$ of mixed strategies in $M$ converging to $m$. Following Reny (1999), say that player $i$ can secure the payoff $\alpha$ at $m \in M$ in the mixed extension of $G$, if player $i$ has a strategy $m_i \in M_i$ such that $u_i(m_i, m'_{-i}) \geq \alpha$ for all $m'_{-i}$ in some neighborhood of $m_{-i}$. We then say that $m_i$ secures the payoff $\alpha$ at $m$ for $i$.

**Definition 2** (Reny 1999). The mixed extension of $G$ is better-reply secure if whenever $(m, u)$ is in $\Gamma$ and $m$ is not an equilibrium, some player $i$ can secure a payoff strictly greater than $u_i$ at $m$ in the mixed extension of $G$. If in addition the securing (mixed) strategies can always be chosen to have finite supports, then the mixed extension of $G$ is finite-support better-reply secure.

For later reference, we state the following result.

**Theorem 1** (Reny 2009). If the mixed extension of $G$ is better-reply secure then $G$ has the finite deviation property. Moreover, if the mixed extension of $G$ is finite-support better-reply secure then $G$ has the finite-support finite deviation property.\(^5\)

### 3 Strategic approximations

Our central definition is the following.\(^6\)

**Definition 3** A strategic approximation of a game $G = (X_i, u_i)_{i=1}^N$ is a countable set of pure strategies $\tilde{X}^\infty = \times_{i=1}^N \tilde{X}_i^\infty$ contained in $X_i$ such that whenever for

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\(^5\) Reny (2009) does not prove the latter result but it follows immediately from Reny’s proof of the former.

\(^6\) We consider finite sets to be countable (see also Royden 1988). Thus, $\tilde{X}_i^\infty$ below may be finite.
each player $i$, $X_i^1 \subseteq X_i^2 \subseteq \cdots$ is an increasing sequence of finite subsets of $X_i$ whose union contains $X_i^\infty$, any limit of equilibria of the sequence of finite games $(X_i^1, u_i)_{i=1}^n, (X_i^2, u_i)_{i=1}^n, \cdots$ is an equilibrium of $G$.

Several examples illustrate the main ideas.\(^7\)

### 3.1 Existence of a strategic approximation

There is one player whose payoff is 1 if he chooses $x = 0$ and is zero if he chooses any other $x \in [0, 1]$. The unique equilibrium is $x = 0$. However, any sequence of finite approximations whose strategy sets become dense in $[0, 1]$ and which do not eventually contain the strategy $x = 0$ admits sequences of equilibria that converge to any element of $[0, 1]$. Consequently, "good" approximations must include the strategy $x = 0$. Moreover, if $X_1, X_2, \ldots$ is any sequence of finite subsets of $[0, 1]$ such that $0 \in X_n$ for $n$ large enough, then the limit of any sequence of equilibria of the finite games converges to $x = 0$. This game therefore possesses a strategic approximation, namely $X^\infty = [0]$.\(^8\)

### 3.2 Nonexistence of a strategic approximation I

The following two-person example is taken from Simon (1987), and we encourage the reader to consult that paper for the details of arguments omitted here. Player 1 chooses $x \in [0, 1]$ and player 2 chooses $y \in [0, 1]$. A player’s payoff is 1 if his choice is equal to one-half of his opponent’s choice and positive or if his choice is 1 and his opponent’s choice is zero. Otherwise, a player’s payoff is zero.

It is not difficult to see that this game possesses no pure strategy equilibrium. It is not much more difficult to see that there are no mixed equilibria in which either player assigns positive probability to any pure strategy. Thus, the only possible equilibria are those in which each player employs an atomless mixed strategy. Finally, any pair of atomless mixed strategies is easily seen to be an equilibrium because given the opponent’s strategy a player receives an expected payoff of zero regardless of the pure strategy he employs.

If one attempts to approximate this infinite game, then regardless of the finite subsets of pure strategies one employs, the approximation will possess a pure strategy equilibrium. Indeed, consider a finite approximation and suppose that player 1’s smallest positive strategy is $x'$ and 2’s is $y'$ where $x' \leq y'$. Then, either $(x', y')$ is an equilibrium or player 1’s finite strategy set contains the strategy $x'/2$ in which case $(y'/2, y')$ is an equilibrium. But if every finite approximation contains a pure strategy equilibrium

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\(^7\) Note that because the set of mixed strategies, $M$, is sequentially compact, the existence of a strategic approximation of $G$ implies that $G$ possesses an equilibrium in mixed strategies. Each finite game in the approximating sequence possesses a mixed strategy equilibrium by Nash (1950), and the sequential compactness of $M$ ensures that some such sequence converges.

\(^8\) In this very special example, equilibria of the approximating games that include $x = 0$ are themselves equilibria of the original game. One cannot hope to obtain this property in general.
then any limit of such equilibria will be pure and so will not be an equilibrium of the infinite game. This game therefore fails to possess a strategic approximation.

3.3 Nonexistence of a strategic approximation II

Consider the following two-person zero-sum game. Player 1 chooses \( x \in [0, 1] \) and player 2 chooses \( y \in [0, 1]^{\infty} \). Player 1's payoff is one if his choice does not match any of the coordinates of 2's choice and is zero otherwise. Clearly, player 1 can guarantee a payoff of 1 if and only if he employs an atomless mixed strategy. Consequently, a strategy pair is an equilibrium if and only if player 1's strategy is mixed and atomless.

Consider the following natural attempt at approximating this game. For \( n = 1, 2, \ldots \) let \( G_n \) be the \( n \)-th approximating game in which player 1's finite pure strategy set is,

\[
X_n = \left\{ \frac{1}{n^2}, \frac{2}{n^2}, \ldots, \frac{n^2}{n^2} \right\},
\]

and player 2's finite pure strategy set is,

\[
Y_n = (X_n)^n \times \{(1, 1, 1, \ldots)\}.
\]

Thus in the \( n \)-th approximating game player 1 has \( n^2 \) strategies and player 2 can choose any of the finitely many vectors in \([0, 1]^{\infty}\) whose first \( n \) coordinates are each a member of player 1's strategy set \( X_n \) and whose remaining coordinates are equal to 1. It is readily verified that all equilibria of \( G_n \) are in mixed strategies of the following form. Player 1 mixes uniformly among the members of \( X_n \) and player 2 mixes among the members of \( Y_n \) so that (i) each member of \( X_n \) is equally likely to be included in the first \( n \) coordinates of the pure strategy realization, and (ii) the first \( n \) coordinates of any member of \( Y_n \) assigned positive probability are distinct. Taking the limit as \( n \to \infty \) yields, upon extraction of a weak*-convergent subsequence if necessary, limit strategies in which player 1 employs Lebesgue measure on \([0, 1]\) and player 2 employs some mixed strategy—i.e., a probability measure on the Borel subsets of \([0, 1]^{\infty}\). Because Lebesgue measure is atomless, this limit is an equilibrium of the original infinite game.

Even though any limit of equilibria of the particular sequence of approximating games \( G_n \) is an equilibrium of the original infinite game, the original game fails to possess a strategic approximation because approximations of it are not robust to the inclusion of additional strategies. For example, providing player 2 with additional strategies in \( G_n \) by instead defining \( Y_n = (X_n)^n \times \{(1, 1, 1, \ldots)\} \) would permit player

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9 Assigning \([0, 1]\) its usual metric and employing the product topology on \([0, 1]^{\infty}\), the strategy spaces are compact and the payoff function is Borel measurable.

10 For example, for each \( k = 0, 1, \ldots, n - 1 \) player 2 can assign probability \( 1/n \) to the vector \( \frac{1}{n^2}(kn + 1, kn + 2, \ldots, kn + n, 1, 1, 1, \ldots) \).

11 Because 2's strategy space is compact in the product topology a weak*-convergent subsequence is guaranteed to exist.
2 to achieve his highest possible payoff of zero in every equilibrium of \( G_n \) by choosing, for example, the pure strategy \( \frac{1}{n^2} (1, 2, \ldots, n^2, 1, 1, 1, \ldots) \), against which player 1 cannot avoid matching one of the coordinates. This strategy for player 2 paired with any strategy for player 1 constitutes an equilibrium of \( G_n \). Moreover, if in each \( G_n \) player 1’s strategy is, for example, pure then no limit of such strategies is an equilibrium of the original game. Every attempt to approximate this infinite game will have a similar defect. Hence, this game fails to possess a strategic approximation because adding additional strategies to any approximation can produce approximating-game equilibria that are far from any of its equilibria.\(^{12}\)

4 Strategic approximations of mixed extensions

As a preliminary step, we apply the definition of a strategic approximation to the mixed extension, \( \tilde{G} = (M_i, u_i)_{i=1}^{N} \) of \( G \). Evidently, a strategic approximation of \( \tilde{G} \) is a countable subset \( M^\infty = \times_{i=1}^{N} M_i^\infty \) of \( M \) such that whenever, for each player \( i \), \( M_i^1 \subseteq M_i^2 \subseteq \cdots \) is an increasing sequence of finite subsets of \( M_i \) whose union is \( M_i^\infty \), any limit of equilibria of the sequence of approximating games \( (M_i^1, u_i)^{N}_{i=1}, (M_i^2, u_i)^{N}_{i=1}, \ldots \) is an equilibrium of \( \tilde{G} \).

A first indication of the relevance of the finite deviation property to the existence of strategic approximations is the following.

**Theorem 2** If \( G \) has the finite deviation property, then the mixed extension of \( G \) has a strategic approximation.

**Proof** If \( G \) has the finite deviation property, then every non equilibrium point of \( G \) is contained in an open set of non equilibrium points. Hence, the set of non equilibrium points of \( G, U \) say, is open. We note that, being an open subset of a compact metric space, every open cover of \( U \) has a countable subcover.\(^{13}\)

By the finite deviation property, for every \( m \in U \) there is a neighborhood \( U^m \) of \( m \) and a finite subset \( D^m \) of \( M \) such that \( D^m \) is a finite deviation set for \( U^m \). This yields a collection of pairs \( (U^m, D^m) \), one pair for each \( m \in U \).

Because the \( U^m \) form an open cover of \( U \), there is a countable subcollection, \( \{(U^1, D^1), (U^2, D^2), \ldots \} \) of \( \{(U^m, D^m)\}_{m \in U} \) such that \( U = \cup_{k=1}^{\infty} U^k \). For each player \( i \) and every \( k = 1, 2, \ldots \) let \( D^i_k \) be the projection of \( D^k \) onto \( M_i \) and define \( M_i^\infty = \cup_{k=1}^{\infty} D^i_k \). It suffices to show that the countable set \( M^\infty = \times_{i=1}^{N} M_i^\infty \) is a strategic approximation of \( \tilde{G} \).

For each player \( i \), let \( M_i^1 \subseteq M_i^2 \subseteq \cdots \) be an increasing sequence of finite subsets of \( M_i \) whose union contains \( M_i^\infty \), and for each \( n \) let \( \mu_n \in \Delta(M) \) be a mixed strategy equilibrium of the finite game \( \tilde{G}^n = (M_i^1, u_i)^{N}_{i=1} \). Thus, for example, if \( m_i \in M_i^1 \), then \( \mu_n^i (m_i) \) is the probability that player \( i \) assigns to \( m_i \) in the equilibrium \( \mu_n \) of

\(^{12}\) An infinite-dimensional strategy space for player 2 is not essential. The same features can be obtained when 2’s strategy space is \([0, 1]\) by using a Peano curve construction mapping \([0, 1]\) onto \( \cup_k [0, 1]^k \) to define payoffs.

\(^{13}\) See, e.g., Dugundji (1989, ch. XI, Th. 7.2).
Suppose that $\mu^u \to \mu^* \in \Delta(M)$. We must show that $\mu^*$ is an equilibrium of $\tilde{G}$.

Let $m^* \in M$ denote the distribution of $X$ of $\mu^*$ and let $m^* \in M$ denote the distribution on $X$ of $\mu^*$. By Lemma 1 in the appendix, $m^u$ converges to $m^*$. We claim that $m^*$ is an equilibrium of $G$. If not, then $m^* \in U$ and there exists $(U^k, D^k)$ such that $m^* \in U^k$ and $D^k \subseteq M$ is a finite deviation set for $U^k$. Because $m^u \to m^*$, we have $m^u \in U^k$ for $n$ large enough, so that for all such $n$ some player $i_n$ can profitably deviate from $m^u$ by employing a strategy $\hat{m}^{n}_{i_n}$ in $D^k_{i_n}$. Consequently, for all $n$ large enough,

$$u_{i_n}(\hat{m}^{n}_{i_n}, \mu^{n}_{-i_n}) = u_{i_n}(\hat{m}^{n}_{i_n}, m^{n}_{-i_n}) > u_{i_n}(m^u) = u_{i_n}(\mu^u),$$

where the first and third lines follow because $(\hat{m}^{n}_{i_n}, \mu^{n}_{-i_n})$ and $\mu^u$—both in $\Delta(M)$—are payoff equivalent (see Sect. 2) to the distributions, $(\hat{m}^{n}_{i_n}, m^{n}_{-i_n})$ and $m^u$ on $X$.

For $n$ sufficiently large, $D^k_{i_n}$ is contained in $M^u_{i_n}$ so that $\hat{m}^{n}_{i_n}$ is a feasible strategy for player $i_n$ in $\tilde{G}_n$. But then $u_{i_n}(\hat{m}^{n}_{i_n}, \mu^{n}_{-i_n}) > u_{i_n}(\mu^u)$ for all $n$ large enough contradicts the fact that $\mu^u$ is an equilibrium of $\tilde{G}^n$. We conclude that $m^* \in M^*$ is an equilibrium of $\tilde{G}$.

But then $\mu^*$ is an equilibrium of $G$ because for every player $i$ and every $m_i \in M_i$,

$$u_i(m_i, \mu^*_{-i}) = u_i(m_i, m^*_{-i}) \\ \leq u_i(m^*) \\ = u_i(\mu^*),$$

where, once again, the first and third lines hold because $(m_i, \mu^*_{-i})$ and $\mu^*$—both in $\Delta(M)$—are payoff equivalent to their distributions $(m_i, m^*_{-i})$ and $m^*$ on $X$, and the second line follows because $m^*$ is an equilibrium of $G$. □

Unfortunately $G$ need not possess a strategic approximation even though its mixed extension does, as the following example demonstrates.15

4.1 Nonexistence of a strategic approximation III

There are three players, Players 1 and 2 participate in a first-price all-pay auction with a uniform tie-break rule in which it is common knowledge that the object at auction

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14 The $m_i$'s are thus treated as pure strategies in the finite games that approximate the mixed extension of $G$ even though they are mixed strategies in the original game $G$.

15 The infinite game in example II does not furnish such an example because its mixed extension does not possess a strategic approximation. To see this, note that for any approximation of its mixed extension one can add a pure strategy to player 1's strategy set that, according to each of the finitely many mixed strategies available to player 2, occurs with probability zero in each coordinate of 2's realized pure strategy. This creates a pure-strategy equilibrium of the approximating game in which player 1 employs the added pure strategy. The limit of such equilibria, being pure for player 1, is not an equilibrium of the infinite game.
is worth one dollar to each of them. Each player can submit a bid from \([0, 1]\) — player 1 chooses \(x\) and player 2 chooses \(y\) — and they are each risk neutral.

Player 3 chooses \(z \in [0, 1]^\infty\) and always receives a payoff of zero regardless of the choices of \(x, y\), and \(z\). If any coordinate of \(z\) is equal to 1’s choice of \(x\), then the outcome of the auction is null and void — i.e., neither player 1 nor player 2 pays his bid and neither player wins the object — and in addition player 1 loses a dollar. Otherwise, player 3’s choice has no effect.

Ignoring for the moment the presence of player 3, a standard argument establishes that there is a unique equilibrium in the all-pay auction, namely that both players 1 and 2 independently randomize according to Lebesgue measure on \([0, 1]\). Given these strategies, player 3’s presence has no effect on expected payoffs because regardless of 3’s choice of \(z\), the probability that one of its coordinates matches 1’s choice of \(x\) is zero.

In fact, every equilibrium of the 3-player game is such that players 1 and 2 choose according to Lebesgue measure, while player 3’s choice of mixed strategy can be arbitrary. This characterization follows from the fact that for any strategy of player 2, player 1 can achieve a payoff arbitrarily close to a best reply in the auction by employing an atomless strategy. Consequently, player 1 can, with probability one, ensure that his choice not match any coordinate of 3’s choice while simultaneously achieving a payoff in the auction that is arbitrarily close to maximal given 2’s strategy. Therefore, in any equilibrium of the 3-player game the results of the auction will stand with probability one and player 1’s strategy must be a best reply in the auction. Since the results of the auction stand with probability one, player 2’s strategy must also be a best reply in the auction and the argument is complete.

For the same reason as in the previous example, this game fails to possess a strategic approximation. Regardless of the attempted strategic approximation, adding to player 3’s pure strategy set a pure strategy whose coordinates contain every pure strategy available to player 1 creates an equilibrium in which player 3 chooses that strategy and players 1 and 2 each choose any pure strategy. Limits of such strategies, being pure for both players 1 and 2, are not equilibria of the original game.

Finally, it can be shown that this game’s mixed extension is better-reply secure and so by Theorem 1 this game has the finite deviation property. Consequently, the mixed extension of this game (by Theorem 2), but not the game itself, possesses a strategic approximation.\(^{16}\) Our next result shows that the discrepancy is due to the failure of the game to possess the finite-support finite deviation property.

### 5 Strategic approximations of G

Our first result on the existence of strategic approximations of \(G\) follows the same line of reasoning as in the proof of Theorem 2.

\(^{16}\) The example neither relies on three players nor on infinite-dimensional strategy spaces. All of its features can be obtained with 2 players and pure strategy sets \([0, 1]\) and \([0, 1]^2\) where player 2’s second coordinate is payoff-irrelevant for him and is mapped onto \(U_1[0, 1]^2\) using the Peano curve construction. The image of 2’s second coordinate is used against player 1 as in the three-player game.
Theorem 3 If $G$ has the finite-support finite deviation property, then $G$ has a strategic approximation.

Proof Maintain the same notation (i.e., $U$, $U^m$, $D^m$, etc.) and construct $M^\infty$ as in the proof of Theorem 2. By the finite-support finite deviation property, we may assume that each member of $M^\infty$ has finite support. For each player $i$, let $X_i^{\infty}$ be the countable set that is the union of the supports of all the members of $M_i^\infty$ and let $X^{\infty} = \times_{i=1}^{N} X_i^{\infty}$.

We wish to show that $X^{\infty}$ is a strategic approximation of $G$.

For each player $i$, let $X_i^1 \subseteq X_i^2 \subseteq \cdots$ be an increasing sequence of finite subsets of $X_i$ whose union contains $X_i^{\infty}$, and for each $n$ let $m^n$ be a mixed strategy equilibrium of the finite game $G^n = (X_i^n, u_i)_{i=1}^{N}$ and suppose that $m^n \to m^* \in M$. We must show that $m^*$ is an equilibrium of $G$.

If not, then $m^* \in U$ and there exists $(U^1, D^1)$ such that $m^* \in U^1$ and $D^1 \subseteq M$ is a finite deviation set for $U^1$. Because $m^n \to m^*$, we have $m^n \in U^1$ for $n$ large enough, so that for all such $n$ some player $i_n$ can profitably deviate from $m^n$ by employing a strategy $m_{i_n}^n$ in $D_{i_n}^1$. Moreover, because the union of the supports of the members of $D_{i_n}^1$ is contained in $X_{i_n}^n$ for $n$ large enough, the profitable deviation $m_{i_n}^n$ is eventually feasible for player $i_n$ in $G^n = (X_i^n, u_i)_{i=1}^{N}$. This contradicts the fact that $m^n$ is an equilibrium of $G^n$ and we conclude that $m^*$ is an equilibrium of $G$. □

An immediate consequence of Theorems 1 and 3 is the following.

Corollary 1 If the mixed extension of $G$ is finite-support better-reply secure, then $G$ has a strategic approximation.

Both Dasgupta and Maskin (1986) and Simon (1987) use approximation techniques to establish the existence of an equilibrium. As shown in Reny (1999), their hypotheses imply that the game’s mixed extension is better-reply secure. Reny’s discussion in fact demonstrates that their hypotheses imply finite-support better-reply security. Consequently, the hypotheses of Corollary 1 weaken those of both Dasgupta and Maskin (1986) and Simon (1987), and so their hypotheses too imply the existence of a strategic approximation.\(^{17}\)

Our final two results provide sufficient conditions for a better-reply secure game to be finite-support better-reply secure and hence to have a strategic approximation. The second of the two results is implied by the first, but its hypotheses are simpler to check. The hypotheses of both results are satisfied in many economic games (Bertrand, Hotelling, auctions, etc.).

Observe that for every player $i$ and every $m \in M$, every payoff below $\lim \inf_{m'_{-i} \to m_{-i}} u_i(m_i, m_{-i})$ can be secured at $m$ by $m_i$. If such payoffs can be secured in a particularly “nice” way, then as we now show, a strategic approximation exists.

Theorem 4 Suppose that the mixed extension of $G = (X_i, u_i)_{i=1}^{N}$ is better-reply secure. If for every $m \in M$ and every player $i$, each payoff below $\lim \inf_{m'_{-i} \to m_{-i}}$

\(^{17}\) In fact, the hypotheses of Dasgupta and Maskin (1986) are sufficiently strong to guarantee that for any game satisfying them, any dense subset of the players’ Euclidean strategy spaces is a strategic approximation.
$u_i(m_i, m'_{-i})$ can be secured at $m$ by a mixed strategy for $i$ whose atomless part, for all $x_{-i}$, assigns probability zero to the set of $x_i$ such that $u_i(\cdot)$ is discontinuous at $(x_i, x_{-i})$, then $G$ admits a strategic approximation.

**Proof** For each player $i$ and every $m \in M_i$, define $u_i(m_i, m_{-i}) = \lim_{m'_{-i} \to m_{-i}} u_i(m_i, m'_{-i})$. Consequently, player $i$ can secure the payoff $\alpha_i$ at $m \in M_i$ if and only if $\tilde{u}_i(m_i, m_{-i}) \geq \alpha_i$ for some $m_i \in M_i$.

Fix a player $i$, $m \in M_i$, and $\varepsilon > 0$. By hypothesis there is a strategy $\tilde{m}_i \in M_i$ securing the payoff $\tilde{u}_i(m_i, m_{-i}) - \varepsilon$ at $m$ (i.e., $\tilde{u}_i(\tilde{m}_i, m_{-i}) \geq u_i(m_i, m_{-i}) - \varepsilon$) and such that the atomless part of $\tilde{m}_i$, for all $x_{-i}$, assigns probability zero to the set of $x_i$ such that $u_i(\cdot)$ is discontinuous at $(x_i, x_{-i})$.

Decompose $\tilde{m}_i$ into $\tilde{\nu}_i = \tilde{v}_i + \tilde{\lambda}_i$, where $\tilde{v}_i$ is the atomless part of $\tilde{m}_i$, and $\tilde{\lambda}_i$ is concentrated on a countable set. Assume that both $\tilde{v}_i(X_i)$ and $\tilde{\lambda}_i(X_i)$ are nonzero.\(^{18}\)

Define the probability measures $v_i = \tilde{v}_i / \tilde{v}(X_i)$ and $\lambda_i = \tilde{\lambda}_i / \tilde{\lambda}(X_i)$ on $X_i$. Since the probability measures on $X_i$ with finite support are dense in $M_i$ (see Billingsley 1968), there is a sequence $(v^k_i)_{k=1}^\infty$ of elements of $M_i$ having finite support such that $v^k_i \to v_i$.

For each $x \in X_i$, let $w_i(x) = \liminf_{x' \to x} u_i(x')$. Consequently, $w_i$ is real-valued (since $u_i(\cdot)$ is bounded) and lower semicontinuous on $X$. In addition, because $u_i(x) = u_i(\cdot)$ is continuous at $x$, we have by hypothesis that,

$$v_i \{ x_i : w_i(x_i, x_{-i}) \neq u_i(x_i, x_{-i}) \} = 0, \quad \text{for all } x_{-i} \in X_{-i},$$

so that

$$\int_{X_{-i}} w_i(x_i, x_{-i}) \, dv_i = u_i(v_i, x_{-i}), \quad \text{for all } x_{-i} \in X_{-i}, \quad (1)$$

and the function of $x_{-i}$ defined by both sides of (1) is continuous on $X_{-i}$.\(^{19}\)

Because $(v^k_i, m_{-i})$ converges weakly to $(v_i, m_{-i})$, the lower semicontinuity of $u_i(\cdot)$ implies,

$$\liminf_{k \to \infty} \int_X w_i(x_i, x_{-i}) \, dv^k_i \, dm_{-i} \geq \int_X w_i(x_i, x_{-i}) \, dv_i \, dm_{-i} = u_i(v_i, m_{-i}), \quad (2)$$

where the second line follows from (1). Hence for all $k$ large enough,

---

\(^{18}\) The argument that follows is easily modified to handle the cases in which one of $\tilde{v}_i$ or $\tilde{\lambda}_i$ is zero.

\(^{19}\) To see continuity, note that if $x'_{-i} \to x_{-i}$, then the sequence of functions $u_i(\cdot, x'_{-i})$ on $X_i$ converges to the function $u_i(\cdot, x_{-i})$ pointwise for $v_i$ a.e. $x_i \in X_i$. Since $u_i(\cdot)$ is bounded, Lebesgue's dominated convergence theorem implies $u_i(v_i, x'_{-i}) \to u_i(v_i, x_{-i})$.
\[
    u_i(v_i^k, m_{-i}) = \lim_{m'_{-i} \to m_{-i}} \inf_{m'_i} u_i(v_i^k, m'_i) \\
    \geq \liminf_{m'_{-i} \to m_{-i}} \int_{X} w_i(x_i, x_{-i}) dv_i^k dm'_{-i} \\
    \geq \int_{X} w_i(x_i, x_{-i}) dv_i^k dm_{-i} \\
    \geq u_i(v_i, m_{-i}) - \varepsilon,
\]

where the second line holds because \( m_i(\cdot) \leq u_i(\cdot) \), the third line holds because \( m_i(\cdot) \) is lower semicontinuous, and the fourth holds by (2).

Choose a sequence of finite-support strategies \( \lambda_i^k \in M_i \), such that \( \lambda_i^k(x_i) \to \lambda_i(x_i) \) for each of the countably many \( x_i \) given positive weight by \( \lambda_i \). Because \( u_i(\cdot) \) is bounded,

\[
    u_i(\lambda_i^k, m_{-i}) \geq u_i(\lambda_i, m_{-i}) - \varepsilon, \quad \text{for } k \text{ large enough.}
\]

Letting \( m_i^k = a v_i^k + (1 - a) \lambda_i^k \), where \( a = \bar{v}_i(X_i) \) (so that \( 1 - a = \bar{\lambda}_i(X_i) \)) gives, for \( k \) large enough,

\[
    u_i(m_i^k, m_{-i}) = u_i(a v_i^k + (1 - a) \lambda_i^k, m_{-i}) \\
    \geq a u_i(v_i^k, m_{-i}) + (1 - a) u_i(\lambda_i^k, m_{-i}) \\
    > a(u_i(v_i, m_{-i}) - \varepsilon) + (1 - a)(u_i(\lambda_i, m_{-i}) - \varepsilon) \\
    = \liminf_{m'_{-i} \to m_{-i}} \left[ a u_i(v_i, m'_{-i}) + (1 - a) u_i(\lambda_i, m'_{-i}) \right] - \varepsilon \\
    = u_i(\bar{m}_i, m_{-i}) - \varepsilon \\
    \geq u_i(m_i, m_{-i}) - 2\varepsilon,
\]

where the third line follows from (3) and (4), and the fourth line follows from the continuity of \( u_i(v_i, m'_{-i}) \) in \( m'_{-i} \) established in (1).

We have therefore demonstrated the following. For every player \( i \), every \( \varepsilon > 0 \), and every \( m \in M_i \), there exists \( \bar{m}_i \in M_i \) with finite support such that

\[
    u_i(\bar{m}_i, m_{-i}) \geq u_i(m_i, m_{-i}) - 2\varepsilon,
\]

from which it is immediate that, being better-reply secure, \( G \) is finite-support better-reply secure. The result now follows from Theorems 1 and 3. \qed

Because, as already observed, every payoff below \( \liminf_{m'_{-i} \to m_{-i}} u_i(m_i, m'_{-i}) \) can be secured at \( m \) by \( m_i \), an immediate consequence of Theorem 4 is the following.

**Corollary 2** Suppose that the mixed extension of \( G = (X_i, u_i) \) is better-reply secure. If for each player \( i \) and every \( x_{-i} \in X_{-i} \), \( u_i(\cdot) \) is continuous at \((x_i, x_{-i})\) for all but perhaps countably many \( x_i \in X_i \), then \( G \) admits a strategic approximation.
Remark 1. The countable discontinuity condition in Corollary 2 is strictly weaker than theDasgupta and Maskin (1986) “diagonal discontinuity” restrictions on the set of discontinuities of the players’ payoff functions.

6 Topological perspectives

A rather different view to approximating infinite games might be to insist that every countable dense subset of the players’ strategy spaces constitute a strategic approximation. This point of view evidently attaches special (economic, perhaps) significance to the underlying topology and gives priority to the robustness of the approximation. But if the question is purely to establish a useful sense in which the strategic character of an infinite game can be said not to hinge on the presence of infinitely many strategies, it is unnecessary to insist that every dense subset of strategies constitute a strategic approximation. The existence of a single strategic approximation would seem to suffice and in any case this is the view proposed here.

7 Appendix

Lemma 1. Let $\mu^n$ be a sequence in $\Delta(M)$ converging to $\mu^*$. If $m^n \in M$ is the distribution on $X$ of $\mu^n$ and $m^* \in M$ is the distribution on $X$ of $\mu^*$, then $m^n$ converges to $m^*$.

Proof. Let $f : X \rightarrow \mathbb{R}$ be continuous and bounded. Then,

$$
\int_X f(x) dm^n(x) = \int_M \int_X f(x) dm(x) d\mu^n(m)
$$

$$
\rightarrow \int_M \int_X f(x) dm(x) d\mu^*(m)
$$

$$
= \int_X f(x) dm^*(x),
$$

where the first and third lines follow because any element of $\Delta(M)$ is payoff equivalent to its distribution on $X$ (see footnote 4), and where the limit follows because $\int_X f(x) dm(x)$ is continuous in $m$. \hfill \square

References